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ZEROS OF QUADRATIC FORMS AND THE δ-METHOD

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Abstract

This thesis presents solutions to three problems. First, we show that the optimal covering exponent for the 3-sphere is $\frac{4}{3}$, and this is joint work with T. D. Browning and R. S. Steiner. Next, we prove a result involving $h(-n)$, the class number of an imaginary quadratic field with fundamental discriminant $-n$. We give an asymptotic formula for correlations involving $h(-n)$ and $h(-n-l)$ over fundamental discriminants that avoid the congruence class $1 \pmod 8$. The result is uniform in the shift l, and along the way we also derive an asymptotic formula for correlations between $r_Q(n)$, the number of representations of an integer by a positive definite quadratic form Q. Finally, we study sums of normalised Hecke eigenvalues $\lambda(n)$ of holomorphic cusp forms over thin sequences. Let $F(\boldsymbol{x})$ be a diagonal quadratic form in 4 variables, we give an upper bound for the problem of counting integer solutions of bounded height to $F(\boldsymbol{x}) = 0$ weighted by $\lambda(x_1)$, and as a consequence we derive upper bounds for certain generalised cubic divisor sums. All three problems are solved by counting integer zeros of quadratic forms using the δ-method.

To Shakthi

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Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's *Regulations and Code of Practice for Research Degree Programmes* and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

Signed: ____________________ Date: __________

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Notation

1. As is standard in analytic number theory, for any complex number α, we set $e(\alpha) = \exp(2\pi i\alpha)$, and $e_q(\alpha)$ is used as shorthand for $e(\alpha/q)$.

2. By $\sum_{a \pmod q}^{*}$ we denote restriction to primitive residue classes modulo q, i.e. $(a, q) = 1$.

3. $S(m, n; q) = \sum_{a \pmod q}^{*} e_q(am + \overline{a}n)$ will denote the Kloosterman sum, where $\overline{a}$ is the mutliplicative inverse of a modulo q.

4. $c_q(m) = S(m, 0; q)$ is defined to be Ramanujan's sum.

5. For a smooth real valued function $w : \mathbf{R}^n \to \mathbf{R}$, we denote by $\|w\|_{N,1}$ its L^1 Sobolev norm of order N.

6. $d(n) = \sum_{d|n} 1$ will denote the divisor function.

7. For a prime p, and $u \in \mathbf{Q}^{\times}$, $v_p(u)$ will denote the $p-$adic valuation of u.

8. Unless stated otherwise, $|.|$ will denote the sup norm on $\mathbf{R}^n$.

9. In this thesis we adopt the following convention. All implicit constants that appear in the error terms will be allowed to depend on the underlying quadratic forms. Any further dependence will be indicated by an appropriate subscript.

Chapter 1

Introduction

An integral quadratic form $Q(\boldsymbol{x}) \in \mathbf{Z}[x_1, \ldots, x_r]$ is a homogeneous quadratic polynomial in r variables with integer coefficients. More explicitly, for $1 \leq i, j \leq r$, there exist integers a_{ij} such that $Q(\boldsymbol{x}) = \sum_{1 \leq i,j \leq r} a_{ij} x_i x_j$. Let A be the symmetric matrix associated to Q. The *rank* of Q is defined to be the matrix rank of A, and if A is of full rank, the *discriminant* of Q is defined to be the determinant of A. Quadratic forms arise naturally across mathematics, and the arithmetic of quadratic forms occupies an exalted position within number theory. Perhaps the most fundamental problem is to understand the representation of integers by quadratic forms. Observe that $\boldsymbol{x} = \mathbf{0}$ is always a solution to the equation $Q(\boldsymbol{x}) = 0$, so our aim is to understand non-zero solutions to quadratic forms. Given a quadratic form Q and an integer n, define the set

$$R(n, Q) = \{\boldsymbol{x} \in \mathbf{Z}^r : Q(\boldsymbol{x}) = n, \boldsymbol{x} \neq 0\}.$$

Our problem then translates to asking if $R(n, Q)$ is non-empty. Beginning with Brahmagupta (598-670 CE), who derived a method (see [30]) to find infinitely many solutions to the 'Pell equation' $x^2 - 92y^2 = 1$, this question has led to the development of a deep and beautiful theory.

The central theme of this thesis is counting integer solutions to quadratic forms using a form of the circle method known as the δ-method. We begin with a brief discussion of the Hasse principle and postpone a discussion of the δ-method to Chapter 2.

1.1 Integral Hasse principle

For $R(n, Q)$ to be non-empty, it is clearly necessary that the equation satisfies *local solubility*, i.e. the equation $Q(\boldsymbol{x}) = n$ has a solution in $\mathbf{Z}_p$ for each place $p \leq \infty$ (for $p = \infty$, we regard $\mathbf{Z}_\infty = \mathbf{R}$). It is natural to ask if the converse is true, i.e. does the existence of a non-trivial solution to $Q(\boldsymbol{x}) = n$ in $\mathbf{Z}_p$, for each place p, imply the existence of a non-trivial *global* solution to this equation? If it does, we say that the *Hasse principle* holds.

If $n = 0$, it is sufficient to find a non-trivial rational solution to this equation. Therefore, the well-known Hasse-Minkowski theorem [16, Chapter 6 Theorem 1.1] can be applied to conclude that $R(0, Q)$ is non-empty if and only if $Q(\boldsymbol{x}) = 0$ has a non-trivial solution in $\mathbf{Q}_p$ for each place p (once again, we regard $\mathbf{Q}_\infty = \mathbf{R}$). One might expect that a similar *local-to-global* statement continues to hold for $n \neq 0$. This is not the case, however, as illustrated by the equation

$$x^2 + 17y^2 = 257.$$

Even though the Hasse principle fails in this instance, it does hold in many cases, e.g. if Q is indefinite and $n \geq 4$ ([16, Chapter 9 Theorem 1.5]). As a result, we will now confine ourselves to positive definite quadratic forms. We will give a brief (and partial) survey of some of the main results for such forms. For a more comprehensive account, we recommend the articles of Duke [26] and Schulze-Pilot [80], and the books [16, 61, 54]. If Q is positive definite, by $r(n, Q)$ we denote the cardinality of the set $R(n, Q)$.

Two quadratic forms Q_1 and Q_2 are said to be equivalent over $\mathbf{Z}$ if there exists a unimodular transformation T over $\mathbf{Z}$ such that $Q_1(\boldsymbol{x}) = Q_2(T\boldsymbol{x})$. It is clear that for questions of representations of integers, it suffices to consider quadratic forms that are $\mathbf{Z}-$equivalent. Given the importance of the local behaviour of quadratic forms, we will say that two forms Q_1 and Q_2 are *locally equivalent* if they are equivalent over $\mathbf{Z}_p$ for each place p. Given Q, we define $\operatorname{gen} Q$, the *genus* of Q, to be the equivalence class of forms locally equivalent to Q. We note that if Q is non-singular, then the discriminant is an invariant of $\operatorname{gen} Q$, and since the number of quadratic forms with given discriminant is finite, we see that $\operatorname{gen} Q$ is finite. Moreover, it is known [16, Chapter 9 Theorem 1.3] that if an integer n is locally represented by Q, then there exists $Q^* \in \operatorname{gen} Q$ such that Q^* represents n globally. As a result, if $\operatorname{gen} Q$ consists of a single element, the Hasse principle holds. However, this is not usually the case. But interestingly, $\operatorname{gen} Q = 1$ if Q is the sum of 2, 3 or 4 squares, a fact which allows us to recover the classical results of Fermat, Legendre and Jacobi, respectively.

If the genus of a quadratic form is not composed of only one element, we can nevertheless relate local representation to representations over $\mathbf{Z}$ in an *average* sense. This is the celebrated result of Siegel and Minkowski, which we will state next (see [80] for a very general statement).

Let $O(Q)$ denote the finite group of $\mathbf{Z}$-automorphisms of Q, and let $o(Q)$ denote its cardinality. Define the *genus mass* to be

$$m(\operatorname{gen} Q) = \sum_{Q_j \in \operatorname{gen} Q} \frac{1}{o(Q_j)},$$

and let $w(Q) = \frac{1}{o(Q)m(\operatorname{gen} Q)}$. Then we see that the total mass over the genus is 1. Let

$$r(n, \operatorname{gen} Q) = \sum_{Q_i \in \operatorname{gen} Q} w(Q_i) r(n, Q_i)$$

be the number of weighted representations averaged over the genus.

Theorem. *Let $r \geq 2$ then there exist local densities $\delta_v(n, Q)$ for each place v such that*

$$r(n, \operatorname{gen} Q) = \prod_v \delta_v(n, Q).$$

Moreover, for finite places p we have

$$\delta_p(n,Q) = \lim_{t\to\infty} p^{-t(r-1)} \# \left\{ \boldsymbol{x} \in \mathbf{Z}^n : Q(\boldsymbol{x}) \equiv n \pmod{p^t} \right\}.$$

Although Siegel's theorem illustrates the relationship between local and global representation on average, new ideas are required to answer our fundamental question on the existence of a local-to-global principle for a given form Q. The crucial observation is that $r(n,Q)$ can be viewed as the Fourier coefficients of a certain modular form. Let

$$\Theta(z,Q) = \sum_{\boldsymbol{m}\in\mathbf{Z}^r} e(Q(\boldsymbol{m})z) = \sum_{n=0}^{\infty} r(n,Q)e(nz).$$

It can be shown $\Theta(z,F)$ is a modular form of weight $\frac{r}{2}$ and level N, which is given in terms of the quadratic form, and certain multiplier system (see [54, Theorem 10.8]). The idea of using such a generating series goes back to Hardy and Ramanujan, and the birth of the circle method. This interplay between the circle method and modular forms will be a recurring theme in this thesis.

Using the theory of modular forms, we can write $\Theta(z,F) = E(z) + F(z)$, where $E(z)$ is an Eisenstein series and $F(z)$ is a cusp form. As a result, we can express $r(n,Q) = a_E(n) + a_f(n)$, in terms of the Fourier coefficients of $E(z)$ and $F(z)$ respectively. Building on Siegel's work to give the lower bound $a_E(n) \gg n^{\frac{r}{2}-1}$ and using the 'trivial' bound $a_F(n) \ll n^{\frac{r}{4}}$, Tartakowsky proved the following

Theorem. *Let Q be a positive definite quadratic form in $r \geq 5$ variables. If n is sufficiently large, the local-to-global principle holds for the equation $Q(\boldsymbol{x}) = n$.*

Remark 1.1.1. To establish the bound $a_E(n) \gg n^{\frac{r}{2}-1}$, it is crucial that there is no *anisotropic prime* for Q, i.e. a prime p for which the equation $Q(\boldsymbol{x}) = 0$ has no non-trivial solution in $\mathbf{Q}_p$. If such a prime exists, then $p \mid N$, the level of Q. Moreover, such primes do not exist if $r \geq 5$.

Observe that the trivial bound for $a_F(n)$ is barely insufficient if $r = 4$, and any improvement will yield the Hasse principle. The *Ramanujan-Petersson Conjecture* asserts that

$$a_F(n) \ll n^{\frac{r}{4}-\frac{1}{2}},$$

which, for even r, follows from Deligne's proof [24] of the Weil conjectures. For odd r, this is still open, but Iwaniec [53] and Duke [25] have made improvements over the trivial bound. However, Kloosterman [63] got around this problem by using a variant of the classical Hardy-Littlewood circle method, which we will discuss in greater detail in Chapter 2. To state Kloosterman's result, we need the following definition.

Definition 1.1.2. Let S be a finite set of primes. An integer n is said to have bounded divisibility at S if there exists a constant $k = k(S)$ such that $v_p(n) \leq k$ for each $p \in S$.

Theorem 1.1.3 (Kloosterman)**.** *Let Q be a positive definite quadratic form in* 4 *variables. Then for sufficiently large n with bounded divisibility at the anisotropic primes the local-to-global principle holds for the equation $Q(\boldsymbol{x}) = n$.*

It is interesting to note that using his version of the circle method, Kloosterman [62] also obtained the first improvement on the trivial bound for Fourier coefficients of cusp forms of integral weight.

Remark 1.1.4. Although we have stated Kloosterman and Tartakowsky's theorems qualitatively, they in fact give asymptotic formulae for $r(n, Q)$ with a power saving error term. These formulae are of the shape

$$r(n, Q) = cn^{\frac{r}{2}-1} + O(n^{\frac{r}{2}-1-\delta})$$

for some $\delta > 0$, for a constant c. Moreover, $c \neq 0$ if n satisfies the hypothesis of the respective theorems, and if the equation $Q(\boldsymbol{x}) = n$ has local solutions.

The preceding discussion covers the case of quadratic forms in at least four variables, which brings us to $r = 3$. This is by far the most interesting case, and also the most challenging. Under additional (necessary) hypothesis on n, a statement analogous to Kloosterman's was established by Duke and Schulze-Pilot [29] for positive definite forms. The case of indefinite ternary forms rests on subtle local considerations, and we refer the reader to [80, Section 3].

The problem of representation by quadratic forms can be generalised further to study the representation of quadratic forms *by* quadratic forms. Using ergodic theory, Ellenberg and Venkatesh [32] have established a local-to-global principle if the difference between the ranks of the respective quadratic forms is at least 5. On the other hand, Colliot-Thélène and Xu [18] have shown that many *obstructions* to the Hasse principle in this context can be explained by the Brauer-Manin obstruction.

1.1.0.1 The main counting function

Let Q be a non-singular quadratic form of rank r. If $R(n, Q)$ is known to be infinite, it is natural to investigate the growth of the set $R(n, Q)$. This leads us to define the counting function

$$\mathcal{N}^{(n)}(Q, X) = \{\boldsymbol{x} \in \mathbf{Z}^r : Q(\boldsymbol{x}) = n,\ |\boldsymbol{x}| \leq X\},$$

where $|\boldsymbol{x}|$ is the l^∞ norm on $\mathbf{Z}^r$, and we will be interested in the behaviour of $\mathcal{N}^{(n)}(Q, X)$ as $X \to \infty$. In analogy with $\mathcal{N}^{(n)}(Q, X)$, for $w \in C_0^\infty(\mathbf{R}^r)$, a smooth function with compact support, let

$$N^{(n)}(Q, X) = N_w^{(n)}(Q, X) = \sum_{\substack{\boldsymbol{x} \in \mathbf{Z}^r \\ Q(\boldsymbol{x}) = n}} w\left(\frac{\boldsymbol{x}}{X}\right). \tag{1.1}$$

Heath-Brown [43] has established an asymptotic formula with a power saving error term for $N^{(n)}(Q, X)$ as $X \to \infty$, if $r \geq 4$ and $n \neq 0$. If $n = 0$, then 3 variables suffice. His proof utilises a version of the circle method, known as the δ-method, which bears some resemblance to Kloosterman's. This is also the method used in this thesis to count solutions to quadratic forms. Perhaps the most important aspect of this result is that it delivers a unified treatment of quadratic forms in at least 4 variables (3 variables if $n = 0$). Moreover, the leading constant that appears in Heath-Brown's result is precisely the one[1] that Siegel obtained in his mass formula.

In this dissertation we study three seemingly disparate problems, but the common thread running through each of them is the distribution of integer solutions to quadratic forms, which is accomplished by examining sums of the form $N^{(n)}(Q, X)$ using the δ-method. To illustrate this, we will give a brief overview of our main results.

1.2 Statement of results

In **Chapter 4** we prove a result on the optimal covering exponent for the 3-sphere. For any $r > 0$, let $S^3(r) \subset \mathbf{R}^4$ denote the hypersphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2,$$

of radius r. (We set $S^3 = S^3(1)$ for the unit hypersphere.) In his letter [78] about the efficiency of a universal set of quantum gates, Sarnak has raised the question of how well one can approximate points on S^3 by rational and S-integral points of small height.

Consider the ball $B_\varepsilon(\boldsymbol{\xi}) = \{\boldsymbol{x} \in \mathbf{R}^4 : \|\boldsymbol{x} - \boldsymbol{\xi}\| < \varepsilon\}$, for any $\varepsilon > 0$ and any $\boldsymbol{\xi} \in S^3$, where $\| \cdot \|$ denotes the Euclidean norm on $\mathbf{R}^4$. The spherical cap $S^3 \cap B_\varepsilon(\boldsymbol{\xi})$ has volume $\frac{4\pi}{3}\varepsilon^3 + O(\varepsilon^5)$. Given $r > 0$ such that $r^2 \in \mathbf{Z}$, we let $\lambda(r)$ denote the maximal volume of any cap $S^3 \cap B_\varepsilon(\boldsymbol{\xi})$, for $\boldsymbol{\xi} \in S^3$, which contains no points of the form $\boldsymbol{x}/r$, for $\boldsymbol{x} \in \mathbf{Z}^4$. Sarnak then defines the *covering exponent* to be

$$K(S^3) = \limsup_{r\to\infty} \frac{\log(\#S^3(r) \cap \mathbf{Z}^4)}{\log((\operatorname{vol} S^3)/\lambda(r))}. \tag{1.2}$$

It is easy to see that $\operatorname{vol} S^3 = 2\pi^2$, and by the quantitative version of Theorem 1.1.3 we have $\#S^3(r) \cap \mathbf{Z}^4 = c_r r^2(1 + o(1))$, as $r \to \infty$, for an appropriate (slowly growing) function c_r of r. According to [56, Thm. 20.9] we have $\log r \gg c_r \gg_\epsilon r^{-\epsilon}$, for any $\epsilon > 0$, as long as the largest power of 2 dividing r^2 is bounded absolutely. In particular,

[1] The constant over the infinite place will also depend on the weight function w.

the limit in (1.2) should be understood as running over such r's.

The "big holes" phenomenon, which is described in [78, Appendix 2], shows that $K(S^3) \geq \frac{4}{3}$. Sarnak conjectures that this lower bound should be the truth, before using automorphic forms for PGL_2 to show that $K(S^3) \leq 2$ in [78, Appendix 1]. This upper bound was recovered by Sardari [77] employing the δ-method.

Our main result establishes Sarnak's conjecture for S^3, under the assumption of a natural variant of the Linnik conjecture about sums of Kloosterman sums.

Conjecture 1.2.1 (Twisted Linnik)**.** *Let $B \geq 1$ and let $m, n \in \mathbf{Z}$ be non-zero. Let $k \in \mathbb{N}$ and let $a \in \mathbf{Z}/k\mathbf{Z}$. Then for any $\alpha \in [-B, B]$ we have*

$$\sum_{\substack{c \equiv a (\mathrm{mod}\ k) \\ c \leq X}} \frac{S(m,n;c)}{c} e\left(\frac{2\sqrt{mn}}{c}\alpha\right) \ll_{\epsilon,k,B} (|mn|X)^{\epsilon},$$

for any $\epsilon > 0$.

Building on Sardari's work [77], we present the following result.

Theorem 1.2.2. *Assume the twisted Linnik conjecture. Then $K(S^3) = \frac{4}{3}$.*

Let $F(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2$, and let $\boldsymbol{\xi}$ be a real point on the 3-sphere, i.e. $F(\boldsymbol{\xi}) = 1$. Fix $\delta > 0$ and let $\varepsilon = N^{-\frac{1}{4}+\delta}$. To prove Theorem 1.2.2, it will be sufficient to show that the equation $F(\boldsymbol{x}) = N$ has an integral solution satisfying $|\boldsymbol{\xi} - \boldsymbol{x}/\sqrt{N}| < \varepsilon$. We count the number of such solutions by studying $N^{(l)}(F, X)$ where $l = N$, $X = \sqrt{N}$ and w is a weight function that is localised at the point $\boldsymbol{\xi}$. Historically, this problem has been studied using the Hardy-Littlewood circle method, beginning with the work of Wright [90]. Although Daemen [19] has recently improved Wright's results, his method would only be able to establish a version of Sarnak's conjecture for S^n, with $n \geq 9$.

Chapter 5 explores correlations between class numbers of imaginary quadratic fields. The shifted convolution sum, i.e. the sum $\sum_{m \leq X} a(m)a(m+f)$, where $a(m)$ is an arithmetic function, is among the best-studied in analytic number theory. When the $a(n)$ are Fourier coefficients of automorphic forms (e.g. $d(n)$) information on such correlations can be used to understand properties of their corresponding L-functions. For an overview of the shifted convolution problem, and its applications, we refer the reader to Michel's exhaustive survey article [68, Chapter 4.4].

Let $K = \mathbf{Q}(\sqrt{-n})$ be an imaginary quadratic field and $h(-n) = \#Cl_K$ be its class number. Define the shifted sum

$$D(X,l) = \sum_{1\leq n\leq X}^{\flat} h(-n)h(-n-l), \tag{1.3}$$

where $\flat$ in the above sum denotes restriction to n such that both $-n$ and $-n-l$ are fundamental discriminants, and such that neither is congruent to $1 \pmod 8$. By the class number formula we have $h(-n) = n^{1/2+o(1)}$, and as a result we expect that $D(X,l) \asymp X^{\frac{3}{2}}(X+l)^{\frac{1}{2}}$. Using the δ-method we show that this holds with a power saving error term.

Theorem 1.2.3. *Let $l \geq 0$ be an integer, and let $D(X,l)$ be as above. Let*

$$\delta = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a constant $\widehat{\sigma}(l) = \prod_{p\leq\infty} \sigma_p(l)$ given in (5.23), such that for all $\varepsilon > 0$ the following asymptotic formula holds,

$$D(X,l) = \frac{\widehat{\sigma}(l)}{576} X^{\frac{3}{2}}(X+l)^{\frac{1}{2}} + O_\varepsilon\left(X^{\frac{3}{2}-\frac{1}{30}}(X+l)^{\frac{1}{2}+\frac{3+\delta}{180}+\varepsilon}\right).$$

Moreover, $\widehat{\sigma}(l) \neq 0$ whenever $\sigma_2(l) \neq 0$, and $\widehat{\sigma}(l) \ll 1$, for an implied constant that is independent of l.

A key feature of our result is the uniformity in the shift l. As a result, we see that the main term dominates the error term for $l \ll X^{2-\varepsilon}$. The link to sums of the form (1.1) comes from the following identity of Gauss [17, Proposition 5.3.10],

$$r_3(n) = 12\left(1 - \left(\frac{-n}{2}\right)\right) h(-n). \tag{1.4}$$

We will show that Theorem 1.2.3 then follows from studying a variant of (1.1) with $Q(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2$, and the x_i being subject to certain congruence conditions. This result also yields asymptotic formulae for correlations of $r(n, Q)$ and $r(n + l, Q)$, for positive definite forms Q.

As a particular instance of this, let $r(n) = r(n, x_1^2 + x_2^2) = 4\sum_{d|n} \chi(d)$, where χ is the unique non-principal real character modulo 4, be the number of representations of an integer n as a sum of two squares. For odd l, Iwaniec [55, Theorem 12.5] showed that

$$\sum_{n \le X} r(n)r(n+l) = 8\left(\sum_{d|l} \frac{1}{d}\right) X + O(l^{\frac{1}{3}} X^{\frac{2}{3}}).$$

As a result, the main term dominates the error term when $l \ll X^{1-\varepsilon}$. In our next result we show that the asymptotic formula holds in the wider range $1 \le l \ll X^{\frac{4}{3}-o(1)}$, and we impose no other restrictions on l.

Theorem 1.2.4. *Let $l \ge 1$ be an integer. There exists a constant $c = c(l)$ such that for all $\varepsilon > 0$ we have*

$$\sum_{n \le X} r(n)r(n+l) = cX + O_\varepsilon(X^{\frac{4}{5}}(X+l)^{\frac{3}{20}+\varepsilon}).$$

Finally, in **Chapter 6** we will study averages of Hecke eigenvalues of holomorphic cusp forms over thin sequences. With Heath-Brown's result [43, Theorems 5-7] at hand, we have a clear understanding of the counting function $N^{(n)}(Q, X)$. Therefore, given an arithmetic function $a(m) : \mathbf{N} \to \mathbf{C}$, it becomes incumbent upon us to ask if we can count solutions to $Q(\boldsymbol{x}) = n$ in which one of the variables is weighted by $a(m)$. More precisely, let

$$N^{(n)}(a; X) = \sum_{Q(\boldsymbol{x})=n} w\left(\frac{\boldsymbol{x}}{X}\right) a(x_1), \tag{1.5}$$

where Q and w are as above. For instance, if $a = \Lambda$ then $N^{(n)}(a; X)$ counts weighted solutions to $Q = n$ where one of the co-ordinates is prime. In the non-homogeneous setting, i.e. $l \neq 0$, this problem has been well-studied. Tsang and Zhao [86] showed that every sufficiently large integer $N \equiv 4 \pmod{24}$ can be written in the form $p_1^2 + P_2^2 + P_3^2 + P_4^2$, where p_1 is a prime, and each P_i has at most 5 prime factors.

We will investigate the case where $n = 0$, the $a(m)$ are Fourier coefficients of a holomorphic cusp form, and where not all the variables are weighted. Suppose that a holomorphic cusp form $f(z)$ has Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda(n) n^{\frac{k-1}{2}} e(nz),$$

and then set $a(n) = \lambda(n)$. Let $N(\lambda; X) = N^{(0)}(a; Q)$. Our main result is

Theorem 1.2.5. *Let $Q \in \mathbf{Z}[\boldsymbol{x}]$ be a non-singular diagonal quadratic form in 4 variables, and let w be a smooth function with compact support in $[1/2, 2]^4$. Let $\lambda(n)$ be the normalised Fourier coefficients of a holomorphic Hecke cusp form f of full level and weight k. Then for all $\varepsilon > 0$ we have*

$$N(\lambda; X) \ll_{\varepsilon, f, Q, w} X^{2 - \frac{1}{6} + \varepsilon}.$$

From Heath-Brown's work on estimating $N^{(0)}(Q, X)$ and Deligne's bound for $\lambda(n)$, we obtain the 'trivial' bound $N(\lambda; X) \ll_{\varepsilon} X^{2+\varepsilon}$. Consequently, Theorem 1.2.5 detects cancellation for $\lambda(n)$ along thin sequences. As a consequence of this theorem, we will deduce a result concerning certain cubic divisor sums (see Theorem 6.1.1).

We end our introduction by listing a few problems that we believe are natural extensions to the ones considered in this thesis.

1.3 Directions for future work

Given the implications of Theorem 1.2.2 to quantum computing (see Remark 4.1.3), it would be desirable to unconditionally show that the covering exponent $K(S^3)$ is strictly smaller than 2. It appears difficult to achieve this using the methods developed in Chapter 4. However, Steiner [83, Chapter 6] has proposed an alternative approach that uses the harmonics on S^3 directly. It is also natural to ask if Theorem 1.2.2, as well as Sardari's results, can be generalised to number fields. One might be able to accomplish this using the δ-method of Browning and Vishe [14].

Class numbers and class groups of quadratic fields are an inexhaustible source to draw inspiration from; but we will limit ourselves to pointing out only one avenue for research connected with Chapter 5. Let $h_3(n)$ denote the cardinality of the 3-part of Cl_K, where K is a quadratic field with fundamental discriminant n. It is expected that $h_3(n) \ll n^{\varepsilon}$, but the best bound that is currently known, due to Ellenberg and Venkatesh [31], is

$$h_3(n) \ll n^{\frac{1}{3}+\varepsilon}.$$

However, Heath-Brown and Pierce [47, Corollary 1.4] have shown that

$$\sum_{1 \leq n \leq X} h_3(\pm n)^2 \ll_{\varepsilon} X^{\frac{23}{18}+\varepsilon}.$$

Observe that this is superior to the bound we obtain from applying the best pointwise bound in conjunction with the classical result of Davenport and Heilbronn. It would be interesting to prove a similar result for the shifted sum,

$$\sum_{1 \leq n \leq X} h_3(-n) h_3(-n-l).$$

In light of Getz's recent work [36] recasting Heath-Brown's results [43] in terms of the adelic δ-method, it would be fruitful to obtain a proof of Theorem 1.2.5 using this method. Since Voronoi summation is better behaved adelically, one could speculate that this would lead to improved error terms, and that generalisations to forms of higher level ought to become easier.

It is also natural to seek a generalisation of Theorem 1.2.5 with two arithmetic weights, i.e. a sum of the form

$$\sum_{\boldsymbol{x} \in \mathbf{Z}^4 : Q(\boldsymbol{x})=0} a(x_1) a(x_2) w(P^{-1}\boldsymbol{x}).$$

It will be very interesting to present an asymptotic formula with a power saving error term for this sum, which would in turn give an asymptotic formula for the divisor sum

$$\sum_{m,n\leq X} d(L_1(m,n)L_2(m,n)Q(m^2+n^2)),$$

where L_i are linear forms and Q is an irreducible quadratic form. Using the geometry of numbers approach of Daniel [20], de la Bretéche and Browning [21] gave an asymptotic formula for the aforementioned sum; however it does not give a power saving error term. If our method were to be successfully adapted to this more complicated case, it would produce a power saving error term, and this is currently work in progress [64]. The leading constant that appears in [21] has a geometric interpretation à la Peyre [75], and it would be particularly interesting to see if the constants that appear in the lower order terms also admit a similar geometric interpretation.

Finally, we mention the following problem. Consider the sum

$$\sum_{\substack{\boldsymbol{x}\in\mathbf{Z}^r\\ Q_2(\boldsymbol{x})=0}} \lambda(Q_1(\boldsymbol{x})),$$

for Q_1 and Q_2 non-singular quadratic forms in r variables. This is the cuspidal analogue of the sum considered in [11, Theorem 1]. One possible way to approach it would be to use the δ-method to detect the equation $Q_2(\boldsymbol{x})=0$, as we have done in this thesis, and to use the Petersson trace formula to decompose $\lambda(Q_1(\boldsymbol{x}))$ in terms of Poincaré series, as in the work of Blomer [4]. Blomer also uses a version of the Kuznetsov trace formula for half-integer weight forms, and it would be a challenging task to execute the details in this setting.

Chapter 2

The δ-method

Let $\{a_m\}$ and $\{b_n\}$ be arithmetic sequences, and suppose that we are interested in studying correlations of the form $\sum_{n\leq N} a_n b_n$. Very often, it will be advantageous to separate the summands, and to write $\sum_{m,n\leq N} a_m b_n \delta(m-n)$, where for a real number x,

$$\delta(x) = \begin{cases} 1 & x = 0, \\ 0 & \text{otherwise} \end{cases}$$

denotes the Kronecker δ-symbol. The δ-method consists of writing $\delta(n)$ in terms of certain 'harmonics' and analysing the resulting sums. For instance, we have the following identity,

$$\delta(n) = \frac{1}{q} \sum_{b \pmod q} e_q(bn), \tag{2.1}$$

whenever $q > |n|$, and n is an integer. In this chapter we will give an overview of such expansions and some of their applications in analytic number theory.

2.1 The δ-method and quadratic forms

The problems considered in this thesis are ultimately solved by counting zeros of quadratic forms in bounded domains. The circle method has been particularly efficacious in studying this topic. Perhaps the most well-known form of this method is the Hardy-Littlewood circle method. We begin by giving a brief account of this method in the context of solving diophantine equations, and for a more comprehensive treatment we refer the reader to [87].

Given a degree d form $P(x_1, \dots, x_n)$ with integer coefficients, i.e. a homogeneous polynomial of degree d in n variables, we are interested in estimating the set

$$\mathcal{N}(P,B) = \#\left\{\boldsymbol{x} \in \mathbf{Z}^n : P(\boldsymbol{x}) = 0, |\boldsymbol{x}| \leq B\right\}, \tag{2.2}$$

as $B \to \infty$. Probabilistic considerations lead us to expect that $\mathcal{N}(P,B) \ll_\varepsilon B^{n-d+\varepsilon}$. The celebrated result of Birch [2] uses the Hardy-Littlewood circle method to establish such a bound (an asymptotic formula, in fact) whose main term is of the expected size provided that $n - \dim(P^*) > (d-1)2^d$, where $\dim(P^*)$ is the dimension of the singular locus of the hypersurface $P = 0$. We should note, however, that there are cases where the expectation $\mathcal{N}(P,B) \ll_\varepsilon B^{n-d+\varepsilon}$ can fail (e.g. if the hypersurface $P = 0$ contains a linear subspace of co-dimension 1, we have $\mathcal{N}(P,B) \gg B^{n-2}$). The *Dimension Growth Conjecture* [8, Chapter 3] states that $\mathcal{N}(P,B) \ll_\varepsilon B^{n-2+\varepsilon}$ for any $\varepsilon > 0$.

An application of the Hardy-Littlewood circle method begins with the identity $\delta(x) = \int_0^1 e(\alpha x)\, d\alpha$, and involves expressing $\mathcal{N}(P,B)$ as an integral,

$$\mathcal{N}(P,B) = \int_0^1 S(\alpha)\, d\alpha, \tag{2.3}$$

where

$$S(\alpha) = \sum_{|\boldsymbol{x}| \leq B} e(\alpha P(\boldsymbol{x})). \tag{2.4}$$

The next step is to split the unit interval into 'major' and 'minor arcs'. The major arcs are disjoint unions of intervals centred around rational numbers with 'small' denominator,

$$\mathfrak{M} = \bigcup_{q \leq B^\Delta} \bigcup_{\substack{1 \leq a \leq q \\ (a,q)=1}} [\frac{a}{q} - B^{-d+\Delta}, \frac{a}{q} + B^{-d+\Delta}],$$

for some parameter $\Delta < 1$. The minor arcs $\mathfrak{m}$ are defined to be the complement of

$\mathfrak{M}$ in the interval $[0, 1]$. Over the major arcs the integral

$$\int_{\mathfrak{M}} S(\alpha)\, d\alpha = \sum_{q \leq B^{\Delta}} \sum_{a (\text{mod } q)}^{*} \int_{-B^{-d+\Delta}}^{B^{-d+\Delta}} S\left(\frac{a}{q} + \alpha\right) d\alpha$$

can be evaluated asymptotically using the Euler-Maclaurin formula, and this yields the main term for $\mathcal{N}(P, B)$. However, the analysis over the minor arcs is more intricate, and the aim is to show that $\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha$ is smaller than the contribution from the major arcs. To carry out this analysis, Weyl differencing or van der Corput's method is used to obtain pointwise bounds on $S(\alpha)$. The Hardy-Littlewood method is extremely versatile, and it has been successfully applied to tackle a wide range of diophantine problems (see [10]).

To motivate the δ-method, we list two well-known diophantine problems that cannot be solved using the above version of the Hardy-Littlewood circle method.

1. Establishing the Hasse principle for the representation of an integer n by a positive-definite quadratic form in 4 variables, and

2. Establishing the Hasse principle for cubic forms in at least 10 variables.

The former is, of course, a generalisation of Lagrange's theorem, and for diagonal forms and large enough n, it was first resolved by Kloosterman [63], as we saw in the Introduction. The latter is a celebrated theorem of Heath-Brown [42] in the case of non-singular cubic forms. Although the two results are proved using a form of the circle method, i.e. the proofs begin with an identity similar to the one in (2.3), they differ in the following way. In Kloosterman and Heath-Brown's versions of the circle method, there are no minor arcs. Instead, the unit interval is broken up using a Farey dissection of order Q (see for instance [51, Chapter 1]), leading to sums of the kind

$$\sum_{q \leq Q} \sum_{a (\text{mod } q)}^{*} \int S\left(\frac{a}{q} + \alpha\right) d\alpha,$$

for α lying in the Farey arc,

$$\left(\frac{a}{q} - \frac{1}{q(q+q')}, \frac{a}{q} + \frac{1}{q(q+q'')}\right],$$

where q' and q'' are determined by the conditions $Q-q < q', q'' \leq Q$, $aq' \equiv 1 \pmod q$ and $aq'' \equiv -1 \pmod q$. The next step is to apply the Poisson summation formula, which results in the appearance of complete exponential sums, which need to be estimated. It is in this context that Kloosterman's eponymous sum first appeared, and obtaining a bound of the form $S(m,n;q) \ll q^{1-\delta}$, for some $\delta > 0$, was essential to Kloosterman's proof. In other contexts, we encounter more general exponential sums that can be estimated using techniques developed by Deligne in his proof of the Weil conjectures.

Moreover, both results make use of the *Kloosterman refinement*, i.e. obtaining non-trivial cancellation over the a-sum. We refer the reader to [56, Chapter 20 §3] for an interesting discussion on Kloosterman's circle method, and for a proof of Kloosterman's result [63] using the following expansion of the δ-symbol,

$$\delta(n) = 2\Re \int_0^1 \sum_{q \leq Q} \sideset{}{^*}\sum_{Q < a \leq q+Q} (aq)^{-1} e\left(n\frac{\overline{a}}{q} - \frac{nx}{aq}\right) dx, \tag{2.5}$$

where $Q \geq 1$ is any real number.

Remark 2.1.1. It is sometimes possible to sum non-trivially over the moduli q, and this is often referred to as a *double Kloosterman refinement*. This idea was first used by Hooley [50] in his work on Waring's problem in 7 variables. A double Kloosterman refinement is also executed by Heath-Brown [43, Theorem 4] in his examination of quadratic forms with square discriminant in 4 variables.

The identity (2.5) is much in the spirit of the δ-method that is used in this thesis, which we will now describe.

2.1.1 The δ-method of Duke, Friedlander and Iwaniec

In their influential work on the subconvexity problem for GL_2 automorphic L-functions, Duke, Friedlander and Iwaniec [27] gave a certain expansion of $\delta(n)$ in terms of Ramanujan sums. Their starting point is the identity (2.1), which we can rewrite as $\delta(n) = \frac{1}{q}\sum_{d|q} c_d(n)$. Observe that to detect the linear equation $m = n$, we must take $q \asymp |m|$, whereas in Kloosterman's method, it suffices to take $Q \asymp |m|^{\frac{1}{2}}$. Moreover, Kloosterman's method has the advantage of averaging over the moduli q. On the other hand, Kloosterman's method is hampered by the appearance of the *Kloosterman fraction*, the exponential factor $e_q(\overline{a}n)$, appearing in the expansion, which can be cumbersome for applications. The δ-method of Duke, Friedlander and Iwaniec is in some sense a synthesis of these two identities.

Let $w(x)$ be a smooth, non-negative function such that $w(0) = 0$ and $\sum_{q=1}^{\infty} w(q) = 1$. Observe that

$$\delta(n) = \sum_{q|n} \left(w(q) - w\left(\frac{|n|}{q}\right)\right), \tag{2.6}$$

which is reminiscent of Dirichlet's hyperbola method in the context of the divisor function (see also [41, Page 419] and especially [67, Equation (1)]). Using additive characters to detect the condition $n \mid q$, Duke, Friedlander and Iwaniec showed the following result.

Theorem 2.1.2. *There exists a function $\Delta_q(u) : \mathbf{R} \to \mathbf{R}$ such that*

$$\delta(n) = \sum_{q=1}^{\infty} c_q(n)\Delta_q(n) \tag{2.7}$$

and

$$\Delta_q(u) = \sum_{r=1}^{\infty} (qr)^{-1}(w(qr) - w(|u|/qr)).$$

Proof. Let $\delta_k(n) = \left(w(k) - w\left(\frac{|n|}{k}\right)\right)$. Then we have

$$\begin{aligned}
\delta(n) &= \sum_{q=1}^{\infty} \frac{1}{q} \sum_{r (\mathrm{mod}\ q)} e_q(rn)\delta_q(n) \\
&= \sum_{q=1}^{\infty} \frac{1}{q} \sum_{d|q} \sideset{}{^*}\sum_{r (\mathrm{mod}\ d)} e_d(rn)\delta_d(n) \\
&= \sum_{d=1}^{\infty} c_d(n) \sum_{e=1}^{\infty} (de)^{-1}\delta_{de}(n).
\end{aligned}$$

This completes the proof. □

It is easy to see that $\Delta_q(u)$ is smooth, and its properties have been studied in detail in [28, Section 3]. In practice, to detect the condition $n = 0$ for $|n| < N/2$, w is taken to be a smooth bump function supported in a dyadic range $[Q/2, Q]$, for some $Q \geq 1$. The optimal choice for Q is then seen to be $N^{\frac{1}{2}}$ and consequently, the q sum above is restricted to lie in the range $q \leq 2\sqrt{Q}$. The parameter Q is very important, and it is sometimes called the *conductor* in the subject.

Suppose that we are interested in a binary additive sum of the form $\sum_{n \leq N} a_n b_{n+l}$ where $\{a_n\}$ and $\{b_n\}$ are arithmetic sequences and $l \in \mathbf{Z}$. Using the δ-method we get

$$\sum_{n \leq N} a_n b_{n+l} = \sum_{q \leq 2\sqrt{N}} \sideset{}{^*}\sum_{a (\mathrm{mod}\ q)} e_q(-al) \left(\sum_{m \leq N} a_m e_q(am)\right) \left(\sum_{n \leq N} b_n e_q(-an)\right) \Delta_q(m-n-l).$$

The method is particularly powerful if we have a good understanding of the sequences $\{a_m\}$ and $\{b_n\}$ in arithmetic progressions.

Remark 2.1.3. Ramanujan [76] gave expansions for many functions of arithmetic interest (e.g. the divisor function, or the von Mangoldt function) in terms of Ramanujan sums, and they can sometimes serve as a substitute for the δ-method, as in Young's analysis of the fourth moment of Dirichlet L-functions (see [91, Lemma 5.4]), for example.

Heath-Brown's variant of the δ-method

In this thesis, we will use the following version of (2.7) that is due to Heath-Brown [43, Theorem 1], who used it to establish the Hasse principle for quadratic forms in at least three variables.

Theorem 2.1.4. *There exists an infinitely differentiable function $h : \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$ such that for any $Q \geq 1$,*

$$\delta(n) = c_Q Q^{-2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a (\bmod q)} e_q(an) h\left(\frac{q}{Q}, \frac{n}{Q^2}\right),$$

where $c_Q = 1 + O_A(Q^{-A})$. The function $h(x, y)$ vanishes unless $x \leq \min(1, 2|y|)$ and its derivatives satisfy the bound

$$\frac{\partial^{a+b}}{\partial x^a y^b} h(x, y) \ll_N x^{-1-a-b} \left(x^N + \min\left(1, \frac{x}{|y|}\right)^N\right). \tag{2.8}$$

For a smooth weight function $w \in C_0^\infty(\mathbf{R}^n)$, we define

$$N(P, B) = \sum_{\substack{\boldsymbol{x} \in \mathbf{Z}^n \\ P(\boldsymbol{x})=0}} w(B^{-1}\boldsymbol{x})$$

to be the weighted analogue of $\mathcal{N}(P, B)$. Using Theorem 2.1.4 to detect the equation $P(\boldsymbol{x}) = 0$ we see that

$$\begin{aligned} N(P, B) &= c_Q Q^{-2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a (\bmod q)} \sum_{\boldsymbol{x} \in \mathbf{Z}^n} e_q(aP(\boldsymbol{x})) w(B^{-1}\boldsymbol{x}) h\left(\frac{q}{Q}, \frac{P(\boldsymbol{x})}{Q^2}\right) \\ &= c_Q Q^{-2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a (\bmod q)} \sum_{\boldsymbol{b} (\bmod q)} e_q(aP(\boldsymbol{b})) \sum_{\boldsymbol{x} \equiv \boldsymbol{b} (\bmod q)} w(B^{-1}\boldsymbol{x}) h\left(\frac{q}{Q}, \frac{P(\boldsymbol{x})}{Q^2}\right). \end{aligned}$$

Since $P(\boldsymbol{x})$ is typically of size B^d, the δ-method is applied with $Q = B^{\frac{d}{2}}$. As in Kloosterman's work, the next step is to apply the Poisson summation formula (Lemma 3.2.1) to the innermost sum in each of the summands x_i. Since the length of each x_i-sum is at most B, and the size of the modulus is typically Q, Poisson's formula is useful only when $d < 4$: for $d \geq 4$, the typical size of the modulus exceeds the square of the length of summation. As a result, the δ-method is most suited to studying problems involving quadratic and cubic forms. Nevertheless, for a form of any degree d we get the following expression in terms of complete exponential sums and certain exponential integrals,

$$N(P,B) = c_Q B^{n-d} \sum_{q=1}^{\infty} q^{-n} \sum_{\boldsymbol{c} \in \mathbf{Z}^n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}),$$

where

$$\begin{aligned} S_q(\boldsymbol{c}) &= \sideset{}{^*}\sum_{a (\mathrm{mod}\ q)} \sum_{\boldsymbol{b} (\mathrm{mod}\ q)} e_q(aP(\boldsymbol{b}) + \boldsymbol{b}.\boldsymbol{c}) \\ I_q(\boldsymbol{c}) &= \int_{\mathbf{R}^n} w(\boldsymbol{x}) h\left(\frac{q}{Q}, P(\boldsymbol{x})\right) e_{q/Q}(-\boldsymbol{c}.\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned}$$

Exercising more care, however, Marmon and Vishe [66] have used the δ-method to establish the Hasse principle for smooth quartic forms in at least 28 variables, thereby improving on a previous result of Hanselmann [40], which required at least 40 variables. Also, the expansion in Theorem 2.1.4 has been generalised to number fields by Browning and Vishe [14], and recently Getz [35, 36] has developed an adelic version of the δ-method to study integer solutions to quadratic forms weighted by a smooth function with compact support in the spirit of [43, Theorem 2]. Using an adelic form of the Poisson summation formula, he establishes an asymptotic formula for the weighted analogue of (2.2), with a second order main term (which Heath-Brown's method could not obtain).

Although the δ-method has proved to be quite versatile and effective, we should point out some of its limitations, especially in the context of treating systems of diophantine equations. This is a well-known facet of the method, and we refer the reader to [48] for a discussion of this problem and its relation to simultaneous rational approximations. Nevertheless, in a series of beautiful papers, Browning and Munshi [11, 12, 72] have used the mechanism of *level lowering* (i.e. using the arithmetic nature of the problem to lower the size of the conductor, Q) to handle pairs of quadratic forms in various settings. We will now briefly explain their idea. While attributing this idea to Browning and Munshi in this context, we should also point out that such ideas are already implicit in the work of Wright.

Level lowering

In their first article [11], Browning and Munshi consider the problem of establishing the Hasse principle for varieties which arise as the common zero locus certain pairs of quadratic forms. Let Q_1 and Q_2 be quadratic forms in at least 7 variables such that Q_2 is non-singular, and suppose that the intersection $Q_1(\boldsymbol{x}) = Q_2(\boldsymbol{x}) = 0$ is also non-singular. Then Browning and Munshi count the number of points on the variety X defined to be the common zero locus of the quadratic forms q_1 and q_2 where

$$
\begin{aligned}
q(x_1, \ldots, x_{n+2}) &= Q_1(x_1, \ldots, x_n) - x_{n+1}^2 - x_{n+2}^2 \\
q(x_2, \ldots, x_{n+2}) &= Q_2(x_1, \ldots, x_n).
\end{aligned}
$$

The counting problem is accomplished by studying the smoothed sum

$$
\sum_{\substack{\boldsymbol{x} \in \mathbf{Z}^n \\ Q_2(\boldsymbol{x})=0}} r(Q_1(\boldsymbol{x})) w(P^{-1}\boldsymbol{x}),
$$

where

$$
r(m) = 4 \sum_{d|m} \chi_4(d),
$$

and χ_4 is the unique non-principal real character modulo 4. Consequently, the problem reduces to counting solutions to $Q_2(\boldsymbol{x})$ modulo d for varying d. At this stage, an application of the δ-method would require the conductor to be of size P. However, Browning and Munshi interpret the condition $Q_2(\boldsymbol{x}) = 0$ in two steps: that $Q_2(\boldsymbol{x}) \equiv 0 \pmod{d}$ and that $Q_2(\boldsymbol{x})/d = 0$. Now if the δ-method is applied for each fixed d, this will have the effect of the conductor dropping to $P/\sqrt{d}$, and since d is typically of P, the saving is significant. Naturally, the resulting analysis is technically intricate, but this is the central idea of the article. It is also interesting to note that a similar conductor lowering idea was also used by Sardari [77] in his work on the covering exponent for S^3, a result we improve upon in Chapter 4.

We end this section by recording some results of a slightly technical nature concerning the h-function in Theorem 2.1.4, since we will make repeated use of them in later chapters. We begin by explicitly describing $h(x,y)$ as in [43, Section 3]. Let $\omega(x) = 4c_0^{-1}w_0(4x-3)$, where

$$w_0(x) = \begin{cases} \exp(-(1-x^2)^{-1}), & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

and $c_0 = \int_{-\infty}^{\infty} w_0(x)\,dx$. Then starting from (2.6) Heath-Brown shows that

$$h(x,y) = \sum_j \frac{1}{xj}\left\{\omega(xj) - \omega(|y|/xj)\right\}.$$

Observe that the function $\Delta_q(u)$ in (2.7) corresponds to the function $h\left(\frac{q}{Q}, y\right)$. We have the following result, which follows from the bounds for the derivatives of $h(x,y)$ recorded in Theorem 2.1.4.

Lemma 2.1.5. *Let $f(y)$ be a smooth function with compact support. Let*

$$p_{r,f}(t) = \int_{-\infty}^{\infty} f(y)h(r,y)e(-ty)\,dy.$$

Then for any $N \geq 0$ we have

$$p_{r,f}(t) \ll \min\left\{1, (rt)^{-N}\right\},$$

where the implied constant depends on N and the derivatives of f.

Proof. This is proved in [43, Lemma 17]. □

In a certain sense, the above result states that the Fourier transform of $h(x,y)$ has polynomial decay. The second result is a simple extension of [43, Lemma 9], which we will make use of in Chapter 4.

Lemma 2.1.6. *Let $f(x)$ be a smooth function with compact support. Then for any $M > 0$ we have*

$$\int_{-\infty}^{\infty} f(y)h(r,y)\,dy = f(0) + O_M(r^M\|f\|_{M,1} + r^{2M-1}\|f\|_\infty)\,dy.$$

Proof. We follow Heath-Brown's proof, and his notation; the only difference is in keeping track of the dependence on the derivatives of f. Let $X = \min\left\{1, r^{\frac{1}{2}}\right\}$. Since $h(r,y) \ll r^{\frac{N}{2}-1}$ for $|y| \geq X$, we find that $\int_{|y|\geq X} f(y)h(r,y) \ll r^{N/2-1}\|f\|_{0,1}$.

We will now focus on the integral $\int_{|y|\leq X} f(y)h(r,y)\,dy$. In the region $|y| \leq X$, expanding $f(y)$ by Taylor's theorem we get $f(y) = P_{2M}(y) + O(\|f\|_{2M+1,1}X^{2M+1})$, where $P_{2M}(y)$ is a polynomial of degree $2M$. By [43, Lemma 4], the error term makes a contribution of $O(r^{-1}X^{2M+2}\|f\|_{2M+1,1})$ to the integral above. Finally, [43, Lemmas 6 and 8] produce the main term $f(0)$ with error terms $O(r^{N-1}\|f\|_\infty) + O(\|f\|_{2M,1}x^N X^{-N})$. Choosing $N = 2M$, we get our result. □

To end this chapter, we present a brief overview of other forms of the δ-method, including Munshi's recent work on the subconvexity problem.

2.2 Other versions of the δ-method

Following Kloosterman's work [63], ideas from the circle method have been applied to the problem of estimating L-functions in the critical strip. This was initiated by Bombieri and Iwaniec in their great paper [6], where they established the bound

$$|\zeta(\tfrac{1}{2} + it)| \ll (|t| + 1)^{\frac{9}{56}+\varepsilon}.$$

The 'convexity' bound, $|\zeta(\frac{1}{2} + it)| \ll (|t| + 1)^{\frac{1}{4}+\varepsilon}$, follows from interpolating the bounds for the zeta function on $\Re(s) = 1 + \delta$ and $\Re(s) = -\delta$, for any $\delta > 0$, via the Phragmén-Lindelöf principle. By a 'subconvexity' bound, we mean a bound of the type $|\zeta(\frac{1}{2}+it)| \ll (1+|t|)^{\frac{1}{4}-\delta}$, for any fixed $\delta > 0$ (see [57]). The Lindelöf hypothesis, which follows from the Riemann hypothesis, states that $|\zeta(\frac{1}{2} + it)| \ll (|t| + 1)^{\varepsilon}$.

Although the Bombieri-Iwaniec method is quite technical, it has been beautifully deconstructed by Huxley and Watt [52, Section 4] who have termed it the 'discrete Hardy-Littlewood method'.

Following the Bombieri-Iwaniec method, Jutila [59] gave a variant using overlapping Farey arcs. In this method, the unit interval $[0, 1]$ is covered by intervals of equal length centred at rationals $\frac{a}{q}$, with the denominators q lying in a certain interval $[Q, 2Q]$. As a consequence of the intervals being of equal length, Jutila's method is approximate in nature. More precisely, Jutila showed the following result.

Theorem 2.2.1. *Let $\mathcal{Q}$ be a non-empty set in $[Q, 2Q]$, with $Q \geq 1$. Let $\delta > 0$ be a real number such that $Q^{-2} \ll \delta \ll Q^{-1}$. Define the function*

$$\tilde{I}_{\mathcal{Q},\delta}(x) = \frac{1}{2\delta L} \sum_{q \in \mathcal{Q}} \sideset{}{^*}\sum_{a \pmod q} \mathbf{1}_{[\frac{a}{q}-\delta, \frac{a}{q}+\delta]}(x)$$

where $\mathbf{1}_S(x)$ is the characteristic function of a set S, and $L = \sum_{q \in \mathcal{Q}} \varphi(q)$. Then

$$\int_0^1 \left|1 - \tilde{I}_{\mathcal{Q},\delta}(x)\right|^2 \ll \frac{Q^{2+\varepsilon}}{\delta L^2}.$$

The advantage of Jutila's method lies in the flexibility in choosing the set $\mathcal{Q}$ (for instance, the set can be chosen to consist of squarefree numbers, or it may be chosen so that each element is coprime to a fixed integer), and the method has been successfully deployed in tackling the subconvexity problem. Jutila's method was recently used by Booker, Milinovic and Ng [7] where they show that

$$L(\tfrac{1}{2} + it, f) \ll (2 + |t|)^{\frac{1}{3}} \log(2 + |t|),$$

where f is a normalised holomorphic Hecke eigenform of arbitrary level. The significance of this result is that it establishes a 'Weyl-type' subconvexity bound for $L(s, f)$, and we will make use of it in Chapter 5. As a further illustration of Jutila's method, we refer to the elegant paper of Munshi [69], where the author proves a 'hybrid' subconvexity bound

$$L(\tfrac{1}{2}+it, f\otimes\chi) \ll (M(1+|t|))^{\frac{1}{2}-\frac{1}{18}+\varepsilon}$$

for the twisted L-function $L(s, f\otimes\chi)$ (see 6.2.1). As with the Bombieri-Iwaniec method, Jutila's circle method can be used to set up a bilinear structure in the moduli set $\mathcal{Q}$.

Building on his work with Browning [11, 12], Munshi has used the idea of level lowering to solve the subconvexity problem for $GL(3)$ L-functions in the t-aspect [70]. A major innovation in Munshi's work is his development of the $GL(2)$ δ-method. All the versions of the circle method that we have encountered so far have relied on additive characters. In his version of the δ-method, Munshi uses Petersson's trace formula [56, Corollary 14.23] to obtain a decomposition of the δ-symbol in terms of Fourier coefficients of modular forms. Using this version of the δ-method, Munshi has established subconvexity results for twists of $GL(3)$ L-functions by a Dirichlet character [71]. We would also like to draw the reader's attention to a recent refinement of the δ-method to study the binary additive sum $\sum_n a_n b_n$. In this method, Munshi introduces an extra variable of summation to induce a bilinear structure on the set of moduli that we average over (see [73, 74]). It would be very interesting to see if a similar idea can be applied to counting points on hypersurfaces.

Chapter 3

Some technical results

In this chapter we gather together some useful results on exponential sums and integrals that will appear in the coming chapters.

3.1 Bessel functions

In Chapter 5 we will encounter Bessel functions, and it will be useful to record some of their properties here. For proofs, we refer the reader to [37, Chapter 8] and to Watson's book [88]. For a complex number ν, Bessel functions are solutions to the second-order differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0.$$

A Bessel function of the first kind, denoted $J_\nu(x)$ are solutions to the above ODE that are finite at $x=0$. They admit the following expansion around the origin,

$$J_\nu(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu},$$

where $|\arg\nu|<\pi$. Let $\nu\geq 1$ be an integer. For $x>0$ and $k\geq 0$, we have

$$\left(\frac{x}{1+x}\right)^k J_\nu(x)\ll_{k,\nu}\frac{x^\nu}{(1+x)^{\nu+\frac{1}{2}}}, \tag{3.1}$$

and we also have the recurrence relation

$$(x^{\nu} J_{\nu}(x))' = x^{\nu} J_{\nu-1}(x). \tag{3.2}$$

For an integer $\nu \geq 2$ define

$$W_{\nu}(x) = \frac{e^{i(\frac{\pi}{2}\nu - \frac{\pi}{4})}}{\Gamma(\nu + \frac{1}{2})} \sqrt{\frac{2}{\pi x}} \int_0^{\infty} e^{-y} \left(y(1 + \frac{iy}{2x}) \right)^{\nu - \frac{1}{2}} dy.$$

By [88, Page 206] for $\nu \geq 2$, we have

$$J_{\nu}(x) = e^{ix} W_{\nu}(x) + e^{-ix} \overline{W_{\nu}(x)}, \tag{3.3}$$

and $e^{ix} W_{\nu}(x)$ is called the *Hankel function*. Moreover, one can verify that for $a \geq 0$ we have

$$x^a W_{\nu}^{(a)}(x) \ll_{a,\nu} x(1+x)^{-\frac{3}{2}}, \tag{3.4}$$

whenever $x \gg 1$.

3.2 Summation formulae

In this thesis, we will use certain Poisson-type summation formulae. We record them below. The following lemma is a standard application of the classical Poisson summation formula.

Lemma 3.2.1. *Let $w(x)$ be a smooth function with compact support. Then*

$$\sum_{m \equiv b (\bmod q)} w\left(\frac{m}{X}\right) = \frac{X}{q} \sum_{m \in \mathbf{Z}} \widehat{w}\left(\frac{m}{q/X}\right) e_q(bm),$$

and $\widehat{w}$ denotes the Fourier transform of w.

Proof. We have $\sum_{m \equiv b \ (\bmod\ q)} w(m/X) = \sum_m w\left(\frac{b+qm}{X}\right)$. Let $u(x) = w\left(\frac{b+qx}{X}\right)$. Applying Poisson's summation formula to $\sum_m u(m)$ we get the result. $\square$

Next, we state Voronoi's summation formula for Fourier coefficients of cusp forms [58, Theorem 1.7].

Lemma 3.2.2. *Let $g(x)$ be a smooth function with compact support, and $\lambda(m)$ be the normalised Fourier coefficients of a holomorphic cusp form of weight k and full level. For $(b,q)=1$ we have*

$$\sum_m \lambda(m)e_q(bm)g(m) = \frac{X}{q}\sum_{m=1}^{\infty}\lambda(m)e_q(-\overline{b}m)\check{g}\left(\frac{m}{q^2/X}\right), \tag{3.5}$$

where

$$\check{g}(m) = 2\pi i^k \int_0^{\infty} g(x)J_{k-1}\left(4\pi\sqrt{xm}\right)\,dx, \tag{3.6}$$

is a Hankel-type transform of g, and $\overline{b}$ is the multiplicative inverse of b modulo q.

3.3 Exponential sums

Let

$$\delta_n = \begin{cases} 0, & \text{if } n \equiv 0 \bmod 2,\\ 1 & \text{if } n \equiv 1 \bmod 2, \end{cases} \qquad \epsilon_n = \begin{cases} 1, & \text{if } n \equiv 1 \bmod 4,\\ i, & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

The following result is recorded in [1, Lemma 3], but it goes back to Gauss.

Lemma 3.3.1. *Define the Gauss sum*

$$\mathcal{G}(s,t;q) = \sum_{b (\mathrm{mod}\ q)} e_q(sb^2+tb).$$

Suppose that $(s,q)=1$. Then

$$\mathcal{G}(s,t;q) = \begin{cases} \epsilon_q\sqrt{q}\left(\frac{s}{q}\right)e\left(-\frac{\overline{4s}t^2}{q}\right) & \text{if } q \text{ is odd},\\ 2\delta_t\epsilon_v\sqrt{v}\left(\frac{2s}{v}\right)e\left(-\frac{\overline{8s}t^2}{v}\right) & \text{if } q=2v, \text{ with } v \text{ odd},\\ (1+i)\epsilon_s^{-1}(1-\delta_t)\sqrt{q}\left(\frac{q}{s}\right)e\left(-\frac{\overline{s}t^2}{4q}\right) & \text{if } 4\mid q. \end{cases}$$

If $(s,q)\neq 1$, $\mathcal{G}(s,t;q)=0$ *unless* $(s,q)\mid t$, *in which case we have*

$$\mathcal{G}(s,t;q)=(s,q)\mathcal{G}\left(\frac{s}{(s,q)},\frac{t}{(s,q)};\frac{q}{(s,q)}\right).$$

3.4 Exponential integrals

In the following version of [43, Lemma 10], which is a form of the 'first derivative' bound for exponential integrals, the dependence on the weight function is made explicit.

Lemma 3.4.1. *Let* $w:\mathbf{R}^n\to\mathbf{R}$ *be a smooth function with compact support. Let* $f(\boldsymbol{x})$ *be a smooth function. Suppose that there is a positive real number* λ, *and a set* $A=\{A_2,A_3,\ldots\}$ *of positive real number such that, for all* $\boldsymbol{x}\in\operatorname{supp}(w)$ *we have*

$$|\nabla f|\geq\lambda$$

and

$$\left|\frac{\partial^{j_1+\ldots+j_n}f(\boldsymbol{x})}{\partial_1^j x_1\ldots\partial^{j_n}x_n}\right|\leq A_j\lambda$$

where $j=j_1+\ldots+j_n\geq 2$. *Then for any* $N>0$ *we have*

$$\int w(\boldsymbol{x})e(f(\boldsymbol{x}))\,d\boldsymbol{x}\ll_{A,N}\|w\|_{N,1}\lambda^{-N}.$$

Proof. Following Heath-Brown's proof, we see that there exists a function w_1 such that $|\partial f(\boldsymbol{x})/\partial x_1|\geq\lambda/2n$, for $\boldsymbol{x}\in\operatorname{supp}(w_1)\subset\operatorname{supp}(w)$ and that

$$\int w(\boldsymbol{x})e(f(\boldsymbol{x}))\,d\boldsymbol{x}\ll_A\int w_1(\boldsymbol{x})e(f(\boldsymbol{x}))\,d\boldsymbol{x}.$$

Since $\|w_1\|_{N,1}\ll\|w\|_{N,1}$ for each $N>0$, we get the lemma by repeated integration by parts in the x_1 variable. $\square$

Next, we prove a version of the 'second derivative' bound that we will need.

Lemma 3.4.2. *Let $w(x)$ have support in $[1/2, 2]$ and let $\Psi(x)$ be a smooth function such that $|\Psi''(x)| \geq c$, for some $c > 0$. Then*

$$\int w(x)e(\Psi(x))\, dx \ll c^{-\frac{1}{2}}.$$

Proof. The proof can be found in Tao's lecture notes [84], and we include it here for the sake of completeness. Since

$$\int w(x)e(\Psi(x))\, dx = -\int_{1/2}^{2} w'(x) \int_{1/2}^{x} e(\Psi(y))\, dy\, dx,$$

it is sufficient to prove the lemma in the unweighted case,

$$\int_{1/2}^{2} e(\Psi(x))\, dx.$$

Let τ be a parameter to be chosen in due course. Observe that by our assumption on Ψ'', $|\Psi'(x)| \geq \tau$ except for an interval of length $O(\tau/c)$. Furthermore, on the remaining portion of the interval $[1/2, 2]$, $\Psi'(x)$ is monotonic. As a result,

$$\int_{1/2}^{2} e(\Psi(x))\, dx \ll \frac{1}{\tau} + \tau/c,$$

by the first derivative test. Choosing τ appropriately completes the proof. $\square$

Next, we state without proof a multidimensional version of the above lemma due to Heath-Brown and Pierce [48, Lemma 3.1].

Lemma 3.4.3. *Let $w(\boldsymbol{u})$ be a smooth function with support in $[-1, 1]^n$. Let $\boldsymbol{\lambda} \in \mathbf{R}^n$, and let Q be a real quadratic form with eigenvalues $\varrho_1, \ldots, \varrho_n$. Define*

$$I(Q; \boldsymbol{\lambda}) = \int_{[-1,1]^n} w(\boldsymbol{u})e(Q(\boldsymbol{u}) - \boldsymbol{\lambda}.\boldsymbol{u})\, d\boldsymbol{u}.$$

We have

$$I(Q; \boldsymbol{\lambda}) \ll \left(\int |\widehat{w}(\boldsymbol{u})|\, d\boldsymbol{u} \right) \prod_{i=1}^{n} \min\left(1, \frac{1}{|\varrho_i|^{\frac{1}{2}}}\right).$$

Chapter 4

Covering exponent for S^3

This chapter is devoted to proving Theorem 1.2.2. The content of this chapter is based on joint work with T. D. Browning and R. S. Steiner, which has appeared online [15].

4.1 Introduction

In this chapter, we give a proof of Sarnak's conjecture for S^3, under the assumption of Conjecture 1.2.1, which we recall below.

Conjecture 4.1.1 (Twisted Linnik)**.** *Let $B \geq 1$ and let $m, n \in \mathbf{Z}$ be non-zero. Let $k \in \mathbf{N}$ and let $a \in \mathbf{Z}/k\mathbf{Z}$. Then for any $\alpha \in [-B, B]$ we have*

$$\sum_{\substack{c \equiv a \pmod{k} \\ c \leq X}} \frac{S(m,n;c)}{c} e\left(\frac{2\sqrt{mn}}{c}\alpha\right) \ll_{\epsilon,k,B} (|mn|X)^{\epsilon},$$

for any $\epsilon > 0$.

For comparison, on invoking the triangle inequality, it follows from Weil's bound for the Kloosterman sum (see (4.5)) that the left hand side has size $O_\epsilon(|mn|^\epsilon X^{\frac{1}{2}+\epsilon})$. The usual *Linnik conjecture* corresponds to taking $\alpha = 0$ in Conjecture 4.1.1. The state of play concerning the case $\alpha = 0$ is discussed in work of Sarnak and Tsimerman [79]. As evidence for Conjecture 4.1.1, Steiner [82] has shown that the unconditional estimates achieved in [79] for $\alpha = 0$ continue to hold for any $\alpha \in \mathbf{R}$ such that $|\alpha| \le 1 - \delta$, for a fixed $\delta > 0$. The case $|\alpha| > 1-\delta$ is also discussed in [82], where the introduced twist cancels out the oscillatory behaviour of the Bessel functions, ultimately leading to slightly weaker estimates. Unfortunately, the unconditional estimates obtained in [82] are not sharp enough to prove that that $K(S^3) < 2$ unconditionally.

For convenience, we recall the statement of Theorem 1.2.2. We have

Theorem 4.1.2. *Assume the twisted Linnik conjecture. Then* $K(S^3) = \frac{4}{3}$.

The proof of this theorem is founded on exploiting extra cancellation in sums of the form

$$\sum_{\substack{q \equiv 1 \bmod 2 \\ q \le Q}} q^{-2} S(r^2, c_1^2 + c_2^2 + c_3^2 + c_4^2; q) e_q(-2r\boldsymbol{c}.\boldsymbol{\xi}) K_q(\boldsymbol{c}),$$

for non-zero vectors $\boldsymbol{c} \in \mathbf{Z}^4$, where $K_q(\boldsymbol{c})$ is a certain 4-dimensional oscillatory integral that is revealed through an examination of (4.7) and (4.8). (There are similar expressions for $q \equiv \{0, 2\} \bmod 4$.) Whereas Sardari brings the modulus sign inside, before invoking Weil's bound to estimate the Kloosterman sum, our goal is take advantage of sign changes in it. There are three key problems in carrying out this plan.

The first two problems arise when using partial summation to remove the factor $q^{-1} e_q(-2r\boldsymbol{c}.\boldsymbol{\xi}) K_q(\boldsymbol{c})$. For typical vectors $\boldsymbol{c}$, the derivative of $e_q(-2r\boldsymbol{c}.\boldsymbol{\xi})$ with respect to q is very large. This deficiency is what lies behind our need to study sums of Kloosterman sums twisted by an exponential factor, as in Conjecture 4.1.1. Similarly, the derivative $\frac{\partial}{\partial q} K_q(\boldsymbol{c})$ is also too large, unless q has exact order of magnitude Q. This presents our second problem. To circumvent this difficulty we shall use stationary phase to get an asymptotic expansion of $K_q(\boldsymbol{c})$, to arbitrary precision, before using partial summation to rid ourselves of each term in the asymptotic expansion separately.

Finally, consider the expression in the left hand side of Conjecture 4.1.1. The third problem comes from a need for complete uniformity in m and n in any unconditional treatment of this sum. In fact, in the present situation, we are faced with the harder *Selberg range*, where $\sqrt{|mn|} > X$. Although Steiner [82] has achieved unconditional bounds that go beyond the Weil bound in certain ranges, these fall short of yielding an unconditional proof that $K(S^3) < 2$. Thus, in our work, we shall be content with showing that the optimal covering exponent is a consequence of our twisted version of Linnik's conjecture.

Remark 4.1.3. As outlined by Sarnak [78], the study of $K(S^3)$ has its roots in the Solovay–Kitaev theorem in theoretical quantum computing. Consider the single qubit gate set $S = \{s_1^{\pm}, s_2^{\pm}, s_3^{\pm}\} \subset \mathrm{SU}(2)$, where

$$s_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}, \quad s_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

This set is symmetric and topologically dense in SU(2). Sarnak defines a covering exponent $K(S)$, which measures how efficiently the free group $\langle S \rangle$ generated by S covers SU(2). It follows from Theorem 4.1.2 that $K(S) = \frac{4}{3}$ under the assumption of the twisted Linnik conjecture.

4.2 Preliminaries

4.2.1 Overview

Let $r \in \mathbf{N}$ such that the power of 2 dividing r is bounded absolutely. Let $N = 4r^2$. Fix a choice of $\boldsymbol{\xi} \in \mathbf{R}^4$ such that $F(\boldsymbol{\xi}) = 1$, where F henceforth denotes the non-singular quadratic form

$$F(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

For any $\varepsilon > 0$, we let

$$S_\varepsilon(N) = \left\{ \boldsymbol{x} \in \mathbf{Z}^4 : F(\boldsymbol{x}) = N, \ \|\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi}\| < \varepsilon \right\}.$$

Our primary objective is to produce a lower bound on ε, in terms of N, which is sufficient to ensure that $S_\varepsilon(N)$ is non-empty. Sardari's work shows that $S_\varepsilon(N) \neq \emptyset$ if

$\varepsilon \gg_\delta N^{-\frac{1}{6}+\delta}$, for any $\delta > 0$. This implies that

$$\lambda(r) \ll_\delta N^{-\frac{1}{2}+\delta} = (2r)^{-1+2\delta},$$

for any $\delta > 0$, whence $K(S^3) \leq 2$ in (1.2). Assuming Conjecture 4.1.1, we shall show that $S_\varepsilon(N) \neq \emptyset$ if $\varepsilon \gg_\delta N^{-\frac{1}{4}+\delta}$, for any $\delta > 0$. This implies that $\lambda(r) \ll_\delta r^{-\frac{3}{2}+2\delta}$, whence $K(S^3) \leq \frac{4}{3}$, as required to complete the proof of Theorem 4.1.2.

4.2.2 Notation

We denote by $\|\cdot\|$ the usual Euclidean norm, so that $\|\boldsymbol{x}\| = \sqrt{F(\boldsymbol{x})}$ on $\mathbf{R}^4$. Throughout our work we reserve $\delta > 0$ for a small positive parameter.

One of the key innovations in Sardari's work [77] concerns the introduction of a new basis given by the tangent space of F at $\boldsymbol{\xi}$ and we proceed to recall the construction here. Let $\mathbf{e}_4 = \boldsymbol{\xi}$. (This is the unit vector in the direction of $\nabla F(\boldsymbol{\xi}) = 2\boldsymbol{\xi}$.) Choose an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ for the tangent space $T_{\boldsymbol{\xi}}(F) = \mathbf{e}_4^\perp$. Recalling that $F(\boldsymbol{\xi}) = 1$, it therefore follows that

$$F(u_1\mathbf{e}_1 + \cdots + u_4\mathbf{e}_4) = F(\boldsymbol{u}),$$

for any $\boldsymbol{u} \in \mathbf{R}^4$. Finally, any vector $\boldsymbol{b} \in \mathbf{R}^4$ can be written $\boldsymbol{b} = \sum_{i=1}^4 \hat{b}_i\mathbf{e}_i$, with $\hat{b}_i = \boldsymbol{b}.\mathbf{e}_i$, for $1 \leq i \leq 4$.

4.2.3 Activation of the circle method

We begin by choosing a smooth function $w_0 : \mathbf{R} \to \mathbf{R}_{\geq 0}$ with unit mass, such that $\mathrm{supp}(w_0) = [-1, 1]$. We will work with the weight function $w : \mathbf{R}^4 \to \mathbf{R}_{\geq 0}$, given by

$$w(\boldsymbol{x}) = w_0\left(\frac{\|\boldsymbol{x} - \boldsymbol{\xi}\|}{\varepsilon}\right) w_0\left(\frac{2\boldsymbol{\xi}.(\boldsymbol{x} - \boldsymbol{\xi})}{\varepsilon^2}\right). \tag{4.1}$$

Let

$$\Sigma(w) = \sum_{\substack{\boldsymbol{x}\in\mathbf{Z}^4\\ F(\boldsymbol{x})=N}} w\left(\frac{\boldsymbol{x}}{\sqrt{N}}\right),$$

for any $N \in 4\mathbf{N}$. Observe that $\Sigma(w)$ is precisely the counting function $N^{(N)}(F,X)$ that was defined in the Introduction.

We want conditions on ε, in terms of N, under which $\Sigma(w) > 0$. Indeed, if $\Sigma(w) > 0$, then there exists a vector $\boldsymbol{x} \in \mathbf{Z}^4$ such that $F(\boldsymbol{x}) = N$ and

$$\|\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi}\| < \varepsilon, \quad |2\boldsymbol{\xi}.(\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi})| < \varepsilon^2.$$

It follows from Sardari's argument that $\Sigma(w) > 0$ if $\varepsilon \gg_\delta N^{-\frac{1}{6}+\delta}$, for any $\delta > 0$. Our goal is to draw the same conclusion provided that $\varepsilon \gg_\delta N^{-\frac{1}{4}+\delta}$.

A few words are in order regarding the inequality $|2\boldsymbol{\xi}.(\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi})| < \varepsilon^2$ that is enshrined in our counting function $\Sigma(w)$. Suppose that $\|\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi}\| < \varepsilon$. Then we may write $\boldsymbol{x}/\sqrt{N} = \boldsymbol{\xi} + \varepsilon\boldsymbol{z}$, with $\|\boldsymbol{z}\| < 1$. Under this change of variables, the inequality $|2\boldsymbol{\xi}.(\boldsymbol{x}/\sqrt{N} - \boldsymbol{\xi})| < \varepsilon^2$ is equivalent to $|2\boldsymbol{\xi}.\boldsymbol{z}| < \varepsilon$, and

$$F(\boldsymbol{x}) - N = N\left(2\varepsilon\boldsymbol{\xi}.\boldsymbol{z} + \varepsilon^2 F(\boldsymbol{z})\right).$$

Thus, we must have $|2\boldsymbol{\xi}.\boldsymbol{z}| < \varepsilon$ when the left hand side vanishes. Moreover it is clear that $F(\boldsymbol{x}) - N \ll \varepsilon^2 N$ for any $\boldsymbol{x}$ such that $w(\boldsymbol{x}/\sqrt{N}) \neq 0$.

One 'level lowering' effect of this is that we are allowed to take

$$Q = \varepsilon\sqrt{N}$$

in the δ-method, rather than $Q = \sqrt{\varepsilon N}$, as might at first appear. By Theorem 2.1.4, we conclude that there exists a constant $c_Q = 1 + O_A(Q^{-A})$, for any $A > 0$, such that

$$\Sigma(w) = \frac{c_Q}{Q^2}\sum_{q=1}^{\infty}\sum_{\boldsymbol{c}\in\mathbb{Z}^4} q^{-4} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}), \tag{4.2}$$

where

$$
\begin{aligned}
S_q(\boldsymbol{c}) &= \sideset{}{^*}\sum_{a \pmod q} \sum_{\boldsymbol{b} \pmod q} e_q\left(a\left\{F(\boldsymbol{b})-N\right\}+\boldsymbol{b}.\boldsymbol{c}\right), \\
I_q(\boldsymbol{c}) &= \int_{\mathbf{R}^4} w\left(\frac{\boldsymbol{x}}{\sqrt{N}}\right) h\left(\frac{q}{Q}, \frac{F(\boldsymbol{x})-N}{Q^2}\right) e_q(-\boldsymbol{c}.\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.
\end{aligned}
\tag{4.3}
$$

Since only values of $q \ll Q$ contribute to $\Sigma(w)$ in (4.2), we may henceforth assume that $Q \geq 1$; viz. $\varepsilon^{-1} \leq \sqrt{N}$.

We shall prove that Conjecture 4.1.1 implies $\Sigma(w) > 0$ if $\varepsilon \gg_\delta N^{-\frac{1}{4}+\delta}$, for any $\delta > 0$. In fact we shall establish an asymptotic formula for $\Sigma(w)$, in which the main term involves a pair of constants σ_∞ and $\mathfrak{S}$. The constant σ_∞ is equal to the weighted real density of points on S^3 and is given explicitly in (4.29). The constant $\mathfrak{S}$ is the usual product of non-archimedean local densities, with value

$$
\mathfrak{S} = \prod_p \sigma_p, \quad \sigma_p = \lim_{k\to\infty} p^{-3k} \#\{\boldsymbol{x} \in (\mathbf{Z}/p^k\mathbf{Z})^4 : F(\boldsymbol{x}) \equiv N \bmod p^k\}. \tag{4.4}
$$

We may now record our main result.

Theorem 4.2.1. *Assume Conjecture 4.1.1. Then, for any $\delta > 0$, we have*

$$
\Sigma(w) = \frac{\varepsilon^3 N \sigma_\infty \mathfrak{S}}{2} + O_\delta\left(\varepsilon^4 N^{1+\delta} + \varepsilon^{\frac{5}{2}} N^{\frac{3}{4}+\delta} + \varepsilon N^{\frac{1}{2}+\delta}\right).
$$

We shall see that $\sigma_\infty \gg 1$ in (4.29). Likewise, as remarked upon by Sardari [77, Remark 1.4], we have $\mathfrak{S} \gg_\delta N^{-\delta}$ for any $\delta > 0$, if the power of 2 dividing N is bounded. Thus Theorem 4.2.1 implies Theorem 4.1.2.

The remainder of the chapter is as follows. In §4.3 we shall explicitly evaluate the sum $S_q(\boldsymbol{c})$ using Gauss sums. Next, in §4.4, we shall study the oscillatory integrals $I_q(\boldsymbol{c})$ using stationary phase. Finally, in §4.5, we shall combine the various estimates and complete the proof of Theorem 4.2.1.

4.3 Gauss sums and Kloosterman sums

In this section we explicitly evaluate the exponential sum $S_q(\boldsymbol{c})$ in (4.3), for $\boldsymbol{c} \in \mathbf{Z}^4$ and relate it to the Kloosterman sum. The latter sum satisfies the well-known Weil bound

$$|S(m,n;c)| \leq d(c)\sqrt{(m,n,c)}\sqrt{c}. \tag{4.5}$$

Recalling that $N \in 4\mathbf{N}$, it will be convenient to write $N = 4N'$ for $N' \in \mathbf{N}$. We have

$$S_q(\boldsymbol{c}) = \sum_{a \bmod q}^{*} e_q(-4aN') \prod_{i=1}^{4} \mathcal{G}(a, c_i; q), \tag{4.6}$$

where

$$\mathcal{G}(s,t;q) = \sum_{b \bmod q} e_q\left(sb^2 + tb\right),$$

for given non-zero integers s, t, q such that $q \geq 1$.

Our analysis of $S_q(\boldsymbol{c})$ now differs according to the 2-adic valuation of q. In each case we shall be led to an appearance of the Kloosterman sum.

Suppose first that $q \equiv 1 \bmod 2$. Substituting Lemma 3.3.1 into (4.6) we directly obtain

$$S_q(\boldsymbol{c}) = q^2 \sum_{a \bmod q}^{*} e_q(-4aN' - \overline{4a}F(\boldsymbol{c})) = q^2 S(N', F(\boldsymbol{c}); q),$$

since $S(A, tB; q) = S(tA, B; q)$ for any $t \in (\mathbf{Z}/q\mathbf{Z})^*$.

If $q \equiv 2 \bmod 4$ then we write $q = 2v$, for odd v. This time we obtain

$$\begin{aligned} S_q(\boldsymbol{c}) &= 2^4 \delta_{c_1c_2c_3c_4} v^2 \sum_{a \bmod q}^{*} e_q(-4aN')e_v(-\overline{8a}F(\boldsymbol{c})) \\ &= 4\delta_{c_1c_2c_3c_4} q^2 S(N', F(\boldsymbol{c})/4; v) \\ &= 4\delta_{c_1c_2c_3c_4} q^2 S(2N', F(\boldsymbol{c})/2; q), \end{aligned}$$

since $4 \mid F(\boldsymbol{c})$, when all the c_i are odd.

If $q \equiv 0 \bmod 4$, it follows from Lemma 3.3.1 that

$$S_q(\boldsymbol{c}) = -4(1-\delta_{c_1})\dots(1-\delta_{c_4})q^2 \sideset{}{^*}\sum_{a \bmod q} e_q(-4aN')e_{4q}(-\overline{a}F(\boldsymbol{c})).$$

Thus, in this case, we find that

$$S_q(\boldsymbol{c}) = \begin{cases} 0 & \text{if } 2 \nmid \boldsymbol{c}, \\ -4q^2 S(N, F(\boldsymbol{c}'); q) & \text{if } \boldsymbol{c} = 2\boldsymbol{c}' \text{ for } \boldsymbol{c}' \in \mathbf{Z}^4. \end{cases}$$

4.4 Oscillatory integrals

Recall the definition (4.3) of $I_q(\boldsymbol{c})$, in which w is given by (4.1). We make the change of variables $\boldsymbol{x} = \sqrt{N}\boldsymbol{x}'$ and $\boldsymbol{x}' = \boldsymbol{\xi} + \varepsilon \boldsymbol{z}$. This leads to the expression

$$\begin{aligned} I_q(\boldsymbol{c}) &= N^2 \int_{\mathbf{R}^4} w(\boldsymbol{x}')\, h\left(\frac{q}{Q}, \frac{F(\boldsymbol{x}')-1}{\varepsilon^2}\right) e_{\frac{q}{\sqrt{N}}}(-\boldsymbol{c}.\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}' \\ &= \varepsilon^4 N^2 e_{\frac{q}{\sqrt{N}}}(-\boldsymbol{c}.\boldsymbol{\xi}) \int_{\mathbf{R}^4} w_0(\|\boldsymbol{z}\|) w_0\left(\frac{2\boldsymbol{\xi}.\boldsymbol{z}}{\varepsilon}\right) h\left(\frac{q}{Q}, \frac{y(\boldsymbol{z})}{\varepsilon}\right) e_{\frac{q}{\varepsilon\sqrt{N}}}(-\boldsymbol{c}.\boldsymbol{z})\,\mathrm{d}\boldsymbol{z}, \end{aligned}$$

where $y(\boldsymbol{z}) = 2\boldsymbol{\xi}.\boldsymbol{z} + \varepsilon F(\boldsymbol{z})$. Let $r = q/Q$ and $\boldsymbol{v} = r^{-1}\boldsymbol{c}$. Then we have

$$I_q(\boldsymbol{c}) = \varepsilon^4 N^2 e_r(-\varepsilon^{-1}\boldsymbol{c}.\boldsymbol{\xi}) I_r^*(\boldsymbol{v}), \tag{4.7}$$

where

$$I_r^*(\boldsymbol{v}) = \int_{\mathbf{R}^4} w_0(\|\boldsymbol{x}\|) w_0\left(\frac{2\boldsymbol{\xi}.\boldsymbol{x}}{\varepsilon}\right) h\left(r, \frac{y(\boldsymbol{x})}{\varepsilon}\right) e(-\boldsymbol{v}.\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{4.8}$$

In particular, we have $I_r^*(\boldsymbol{v}) = O(\varepsilon/r)$, since $h(r,y) \ll r^{-1}$ and the region of integration has measure $O(\varepsilon)$.

4.4.1 Easy estimates

Our attention now shifts to analysing $I_r^*(\boldsymbol{v})$ for $r \ll 1$ and $\boldsymbol{v} \in \mathbf{R}^4$. Let $\boldsymbol{x} \in \mathbf{R}^4$ such that $w_0(\|\boldsymbol{x}\|)w_0(2\boldsymbol{\xi}.\boldsymbol{x}/\varepsilon) \neq 0$. Then

$$\frac{y(\boldsymbol{x})}{\varepsilon} = \frac{2\boldsymbol{\xi}.\boldsymbol{x} + \varepsilon F(\boldsymbol{x})}{\varepsilon} < 2.$$

Put $v(t) = w_0(t/6)$. Then $v(y(\boldsymbol{x})/\varepsilon) \gg 1$ whenever $w_0(\|\boldsymbol{x}\|)w_0(2\boldsymbol{\xi}.\boldsymbol{x}/\varepsilon) \neq 0$. We may now write

$$I_r^*(\boldsymbol{v}) = \frac{1}{r} \int_{\mathbf{R}^4} w_3(\boldsymbol{x}) f\left(\frac{y(\boldsymbol{x})}{\varepsilon}\right) e(-\boldsymbol{v}.\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

where $f(y) = v(y) r h(r, y)$ and

$$w_3(\boldsymbol{x}) = \frac{w_0(\|\boldsymbol{x}\|)w_0(2\boldsymbol{\xi}.\boldsymbol{x}/\varepsilon)}{v(y(\boldsymbol{x})/\varepsilon)}. \tag{4.9}$$

Let $p(t) = \hat{f}(t)$ be the Fourier transform of f. Then Lemma 2.1.5 shows that

$$p(t) \ll_j r(r|t|)^{-j}, \tag{4.10}$$

for any $j > 0$. We may therefore write

$$I_r^*(\boldsymbol{v}) = \frac{1}{r} \int_{\mathbf{R}} p(t) \int_{\mathbf{R}^4} w_3(\boldsymbol{x}) e\left(t\frac{y(\boldsymbol{x})}{\varepsilon} - \boldsymbol{v}.\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.11}$$

Building on this, we proceed by establishing the following result.

Lemma 4.4.1. *Let $\boldsymbol{c} \in \mathbf{Z}^4$, with $\boldsymbol{c} \neq \mathbf{0}$. Then*

$$I_q(\boldsymbol{c}) \ll_j \frac{\varepsilon^5 N^2 Q}{q} \min_{i=1,2,3} \left\{|\hat{c}_i|^{-j}, (\varepsilon|\hat{c}_4|)^{-j}\right\},$$

for any $j > 0$.

This result corresponds to [77, Lemma 6.1]. Since $\max_i |\hat{c}_i| \gg \|\boldsymbol{c}\|$, it follows that

$$I_q(\boldsymbol{c}) \ll_j \frac{\varepsilon^5 N^2 Q}{q} (\varepsilon\|\boldsymbol{c}\|)^{-j},$$

for any $j > 0$. In this way, for any $\delta > 0$, Lemma 4.4.1 implies that there is a negligible contribution to (4.2) from $\boldsymbol{c}$ such that either of the inequalities $\|\boldsymbol{c}\| > N^\delta/\varepsilon$ or $\max_{i=1,2,3}\{|\hat{c}_i|, \varepsilon|\hat{c}_4|\} > N^\delta$ hold. Thus, in (4.2), the summation over $\boldsymbol{c}$ can henceforth be restricted to the set $\mathcal{C}$, which is defined to be the set of $\boldsymbol{c} \in \mathbf{Z}^4$ for which $\|\boldsymbol{c}\| \leq N^\delta/\varepsilon$ and $\max_{i=1,2,3}\{|\hat{c}_i|, \varepsilon|\hat{c}_4|\} \leq N^\delta$. It follows from [77, Lemma 6.3] that $\#\mathcal{C} = O(\varepsilon^{-1}N^{4\delta})$.

Proof of Lemma 4.4.1. We make the change of variables $\boldsymbol{x} = \sum_{i=1}^4 u_i\mathbf{e}_i$ in (4.11). In the notation of §4.2.2, let $\boldsymbol{v} = \sum_{i=1}^4 \hat{v}_i\mathbf{e}_i$, where $\hat{v}_i = \boldsymbol{v}.\mathbf{e}_i$. Then, on recalling (4.9), we find that

$$\begin{aligned}
I_r^*(\boldsymbol{v}) &= \frac{1}{r}\int_{\mathbf{R}} p(t)\int_{\mathbf{R}^4} w_3\left(\sum_{i=1}^4 u_i\mathbf{e}_i\right) e\left(\frac{ty(\sum_{i=1}^4 u_i\mathbf{e}_i)}{\varepsilon} - \boldsymbol{u}.\hat{\boldsymbol{v}}\right) \mathrm{d}\boldsymbol{u}\,\mathrm{d}t \\
&= \frac{1}{r}\int_{\mathbf{R}} p(t)\int_{\mathbf{R}^4} \frac{w_0(\|\boldsymbol{u}\|)w_0(2u_4/\varepsilon)}{v((2u_4+\varepsilon F(\boldsymbol{u}))/\varepsilon)} e\left(H(\boldsymbol{u})\right) \mathrm{d}\boldsymbol{u}\,\mathrm{d}t,
\end{aligned}$$

where $H(\boldsymbol{u}) = \frac{t}{\varepsilon}\{2u_4 + \varepsilon F(\boldsymbol{u})\} - \boldsymbol{u}.\hat{\boldsymbol{v}}$. We have

$$\frac{\partial H(\boldsymbol{u})}{\partial u_i} = \begin{cases} 2tu_i - \hat{v}_i & \text{if } 1 \leq i \leq 3, \\ 2tu_4 - \hat{v}_4 + \frac{2t}{\varepsilon} & \text{if } i = 4. \end{cases}$$

The proof of the lemma now follows from repeated integration by parts in conjunction with (4.10), much as in the proof of [43, Lemma 19]. Thus, when $i \in \{1, 2, 3\}$, integration by parts with respect to u_i readily yields

$$I_r^*(\boldsymbol{v}) \ll_j \frac{\varepsilon}{r}\left\{r|\hat{v}_i|^{1-j} + r^{1-j}|\hat{v}_i|^{1-j}\right\} \ll_j \varepsilon r^{-j}|\hat{v}_i|^{1-j},$$

for any $j > 0$, since $r \ll 1$. Likewise, integrating by parts with respect to u_4, we get

$$I_r^*(\boldsymbol{v}) \ll_j \frac{\varepsilon}{r}\left\{r(\varepsilon|\hat{v}_4|)^{1-j} + r^{1-j}(\varepsilon|\hat{v}_4|)^{1-j}\right\} \ll_j \varepsilon r^{-j}(\varepsilon|\hat{v}_4|)^{1-j}.$$

The statement of the lemma follows on recalling (4.7) and the fact that $\boldsymbol{c} = r\boldsymbol{v}$, with $r = q/Q$. □

4.4.2 Stationary phase

The following stationary phase result will prove vital in our more demanding analysis of $I_q(\boldsymbol{c})$ in the next section.

Lemma 4.4.2. *Let φ be a Schwartz function on $\mathbb{R}^n$ and let $N \geq 0$. Then*

$$\int_{\mathbb{R}^n} e^{i\lambda\|\boldsymbol{x}\|^2}\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} =\lambda^{-\frac{n}{2}}\sum_{j=0}^{N} a_j\lambda^{-j} + O_{n,N}\left(|\lambda|^{-\frac{n}{2}-N-1}\|\varphi\|_{2N+3+n,1}\right),$$

where $\|\cdot\|_{k,1}$ denotes the Sobolev norm on $L^1(\mathbf{R}^n)$ of order k and

$$a_j = (i\pi)^{\frac{n}{2}}\frac{i^j}{j!}\left(\Delta^j\varphi\right)(\mathbf{0}).$$

Proof. We follow the argument in Stein [81, §VIII.5.1]. Using the Fourier transform, we can write the integral as

$$\left(\frac{i\pi}{\lambda}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n} e^{-i\pi^2\|\boldsymbol{\xi}\|^2/\lambda}\widehat{\varphi}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}. \tag{4.12}$$

Next, we split off the first N terms in a Taylor expansion around $\mathbf{0}$, finding that

$$e^{-i\pi^2\|\boldsymbol{\xi}\|^2/\lambda} = \sum_{j=0}^{N}\frac{(-i\pi^2\|\boldsymbol{\xi}\|^2/\lambda)^j}{j!} + R_N(\boldsymbol{\xi}).$$

The main term now comes from integration by parts and Fourier inversion. We are left to deal with the integral involving $R_N(\boldsymbol{\xi})$. We have

$$R_N(\boldsymbol{\xi}) \ll_N \left(\frac{\|\boldsymbol{\xi}\|^2}{|\lambda|}\right)^{N+1}, \tag{4.13}$$

which follows from Taylor expansion when $\|\boldsymbol{\xi}\|^2 \leq |\lambda|$ and trivially otherwise. Moreover,

$$\widehat{\varphi}(\boldsymbol{\xi}) = O_A\left(\|\boldsymbol{\xi}\|^{-A}\|\varphi\|_{A,1}\right), \tag{4.14}$$

for any $A \geq 0$. We split up the remaining integral into two parts: $\|\boldsymbol{\xi}\| \leq 1$ and $\|\boldsymbol{\xi}\| > 1$. For the first part we use (4.13) and (4.14) with $A = 2N+1+n$. Recalling the additional factor $\lambda^{-\frac{n}{2}}$ from (4.12), we get an error term of size

$$O_{n,N}\left(|\lambda|^{-\frac{n}{2}-N-1}\|\varphi\|_{2N+1+n,1}\right).$$

For the second part we use (4.13) and (4.14), but this time with $A = 2N+3+n$. This leads to the same overall error term, but with the factor $\|\varphi\|_{2N+1+n,1}$ replaced by $\|\varphi\|_{2N+3+n,1}$. $\square$

4.4.3 Hard estimates

Having shown how to truncate the sum over $\boldsymbol{c}$ in (4.2), we now return to (4.7) for $\boldsymbol{c} \in \mathcal{C}$ and see what more can be said about the integral $I_r^*(\boldsymbol{v})$ in (4.8), with $r = q/Q$ and $\boldsymbol{v} = r^{-1}\boldsymbol{c}$. Our result relies on an asymptotic expansion of $I_r^*(\boldsymbol{v})$, but the form it takes depends on the size of $\varepsilon|\hat{v}_4|$.

It will be convenient to set $\mathbf{a} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$, in what follows. To begin with, we make the change of variables $\boldsymbol{x} = \sum_{i=1}^4 u_i\mathbf{e}_i$ in (4.8). This leads to the expression

$$I_r^*(\boldsymbol{v}) = \int_{\mathbf{R}^4} w_0(\|\boldsymbol{u}\|)w_0(2u_4/\varepsilon)h\left(r, \frac{2u_4}{\varepsilon} + F(\boldsymbol{u})\right)e(-\boldsymbol{u}.\hat{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{u},$$

where $\hat{v}_i = \boldsymbol{v}.\mathbf{e}_i$ for $1 \le i \le 4$. We now write $y = 2u_4/\varepsilon + F(\boldsymbol{u})$, under which we have

$$u_4 = \frac{1}{\varepsilon}\left(-1 + \sqrt{1+\varepsilon^2\{y - u_1^2 - u_2^2 - u_3^2\}}\right). \tag{4.15}$$

Thus

$$I_r^*(\boldsymbol{v}) = \int_{\mathbf{R}} h(r,y)e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right)T(y)\mathrm{d}y, \tag{4.16}$$

where

$$T(y) = e\left(\frac{\varepsilon\hat{v}_4 y}{2}\right)\int_{\mathbf{R}^3} w_0(\|\boldsymbol{u}\|)w_0(2u_4/\varepsilon)e(-\boldsymbol{u}.\hat{\boldsymbol{v}})\,\frac{\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3}{2/\varepsilon + 2u_4}, \tag{4.17}$$

and u_4 is given in terms of y, u_1, u_2, u_3 by (4.15). In particular, on writing $\boldsymbol{x} = (u_1, u_2, u_3)$, we have $w_0(\|\boldsymbol{u}\|)w_0(2u_4/\varepsilon) = \psi_y(\boldsymbol{x})$, where $\psi_y : \mathbf{R}^3 \to \mathbf{R}_{\ge 0}$ is the weight function

$$\begin{aligned}\psi_y(\boldsymbol{x}) = \;& w_0\left(2\varepsilon^{-2}(-1+\sqrt{1+\varepsilon^2\{y-\|\boldsymbol{x}\|^2\}})\right)\\ &\times w_0\left(\sqrt{\|\boldsymbol{x}\|^2+\varepsilon^{-2}(1-\sqrt{1+\varepsilon^2\{y-\|\boldsymbol{x}\|^2\}})^2}\right).\end{aligned} \tag{4.18}$$

We note, furthermore, that the integral in $T(y)$ is supported on $[-1,1]^3$. Moreover, we have

$$\frac{2u_4}{\varepsilon} = \frac{2}{\varepsilon^2}\left(-1+\sqrt{1+\varepsilon^2\{y-\|\boldsymbol{x}\|^2\}}\right) = y-\|\boldsymbol{x}\|^2+O(\varepsilon^2), \tag{4.19}$$

for any $\boldsymbol{x}$ such that $\psi_y(\boldsymbol{x})\neq 0$. In particular, it follows that

$$\frac{1}{2/\varepsilon+2u_4} = \frac{\varepsilon}{2}\left(1+O(\varepsilon^2)\right) \tag{4.20}$$

in (4.17).

Since $e(z)=1+O(z)$, we invoke (4.15) and (4.19) to deduce that

$$e(-\boldsymbol{u}.\hat{\boldsymbol{v}}) = e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right)e\left(\frac{\varepsilon\hat{v}_4}{2}\|\boldsymbol{x}\|^2-\mathbf{a}.\boldsymbol{x}\right)\left(1+O(|\varepsilon\hat{v}_4|\varepsilon^2)\right), \tag{4.21}$$

where we recall that $\mathbf{a}=(\hat{v}_1,\hat{v}_2,\hat{v}_3)$. Thus, it follows from (4.20) that

$$T(y) = \frac{\varepsilon}{2}\left(1+O(\varepsilon^2+|\varepsilon\hat{v}_4|\varepsilon^2)\right)I(y), \tag{4.22}$$

where

$$I(y) = \int_{\mathbf{R}^3}\psi_y(\boldsymbol{x})e\left(\frac{\varepsilon\hat{v}_4}{2}\|\boldsymbol{x}\|^2-\mathbf{a}.\boldsymbol{x}\right)\mathrm{d}\boldsymbol{x}. \tag{4.23}$$

In what follows it will be useful to record the estimate

$$\int_{\mathbf{R}}\left|r^k y^\ell\frac{\partial^k h(r,y)}{\partial r^k}\right|\mathrm{d}y \ll_\ell r^\ell, \tag{4.24}$$

for any $\ell\geq 0$ and $k\in\{0,1\}$. This is a straightforward consequence of [43, Lemma 5]. The stage is now set to prove the following preliminary estimate for $I_r^*(\boldsymbol{v})$ and its partial derivative with respect to r.

Lemma 4.4.3. *Let $k \in \{0,1\}$. Then*

$$r^{2k}\frac{\partial^k I_r^*(\boldsymbol{v})}{\partial r^k} \ll \frac{\varepsilon(1+\varepsilon^3|\hat{v}_4|)}{\max\{1,(\varepsilon|\hat{v}_4|)\}^{\frac{3}{2}}}N^{\delta}.$$

Proof. Suppose first that $k=0$. An application of Lemma 3.4.3 shows that

$$I(y) \ll \frac{1}{\max\{1,(\varepsilon|\hat{v}_4|)\}^{\frac{3}{2}}},$$

since $\|\hat{\psi}_y\|_1 \ll 1$. The desired bound now follows on substituting this into (4.16) and (4.22), before using (4.24) with $k=\ell=0$ to carry out the integration over y.

Suppose next that $k=1$. Then, in view of (4.16), we have

$$\begin{aligned} r^2\frac{\partial I_r^*(\boldsymbol{v})}{\partial r} = & \int_{\mathbb{R}} r^2\frac{\partial h(r,y)}{\partial r}e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right)T(y)\mathrm{d}y \\ & + \int_{\mathbb{R}} h(r,y)e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right)\widetilde{T}(y)\mathrm{d}y, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} \widetilde{T}(y) &= e\left(\frac{\varepsilon\hat{v}_4 y}{2}\right)\int_{\mathbf{R}^3} w_0(\|\boldsymbol{u}\|)w_0(2u_4/\varepsilon)r^2\frac{\partial}{\partial r}e(-\boldsymbol{u}.\hat{\boldsymbol{v}})\,\frac{\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3}{2/\varepsilon+2u_4} \\ &= e\left(\frac{\varepsilon\hat{v}_4 y}{2}\right)\int_{\mathbf{R}^3}(2\pi i\boldsymbol{u}.\hat{\boldsymbol{c}})w_0(\|\boldsymbol{u}\|)w_0(2u_4/\varepsilon)e(-\boldsymbol{u}.\hat{\boldsymbol{v}})\frac{\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3}{2/\varepsilon+2u_4}. \end{aligned}$$

The contribution from the first integral in (4.25) is satisfactory, since $r \ll 1$, on reapplying our argument for $k=0$ and using (4.24) with $k=1$ and $\ell=0$. Turning to the second integral in (4.25), we recall (4.20) and (4.21). These allow us to write

$$\widetilde{T}(y) = \varepsilon\pi i\left(1+O(\varepsilon^2+|\varepsilon\hat{v}_4|\varepsilon^2)\right)\widetilde{I}(y),$$

where

$$\widetilde{I}(y) = \int_{\mathbb{R}^3}\widetilde{\psi_y}(\boldsymbol{x})e\left(\frac{\varepsilon\hat{v}_4}{2}\|\boldsymbol{x}\|^2-\mathbf{a}.\boldsymbol{x}\right)\mathrm{d}\boldsymbol{x}$$

and

$$\widetilde{\psi_y}(\boldsymbol{x}) = \left(r\mathbf{a}.\boldsymbol{x}+\frac{\hat{c}_4}{\varepsilon}\left(-1+\sqrt{1+\varepsilon^2\{y-\|\boldsymbol{x}\|^2\}}\right)\right)\psi_y(\boldsymbol{x}).$$

Here, the definition of $\mathcal{C}$ implies that $r|\mathbf{a}| = \max\{|\hat{c}_1|, |\hat{c}_2|, |\hat{c}_3|\} \leq N^\delta$ and $\varepsilon|\hat{c}_4| \leq N^\delta$. Thus the L^1-norm of the Fourier transform of $\widetilde{\psi_y}$ is $O(N^\delta)$. Once combined with (4.24) with $k = \ell = 0$, we apply Lemma 3.4.3 to estimate $\widetilde{I}(y)$, which concludes our treatment of the case $k = 1$. $\square$

The case $k = 0$ of Lemma 4.4.3 is already implicit in Sardari's work (see [77, Lemma 6.2]). We shall also need the case $k = 1$, but it turns out that it is only effective when r is essentially of size 1. For general r, we require a pair of asymptotic expansions for $I_r^*(\boldsymbol{v})$, that are relevant for small and large values of $\varepsilon|\hat{v}_4|$, respectively. This is the objective of the following pair of results.

Lemma 4.4.4. *Let $A \geq 0$. Then*

$$I_r^*(\boldsymbol{v}) = \frac{\varepsilon I(0)}{2} + O_A\left(\varepsilon^3(1 + \varepsilon|\hat{v}_4|) + \varepsilon(1 + \varepsilon|\hat{v}_4|)^A r^A\right).$$

Proof. Our first approach is founded on the Taylor expansion

$$e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right) = \sum_{j=0}^{A-1} \frac{(-\pi i\varepsilon\hat{v}_4 y)^j}{j!} + R_A(y),$$

where $R_A(y) \ll_A (\varepsilon|\hat{v}_4 y|)^A$. Since $I(y) \ll 1$, we conclude from (4.16), (4.22) and (4.24) that

$$\begin{aligned} I_r^*(\boldsymbol{v}) = {} & \frac{\varepsilon}{2} \sum_{j=0}^{A-1} \frac{(-\pi i\varepsilon\hat{v}_4)^j}{j!} \int_{\mathbf{R}} y^j h(r, y) I(y) \mathrm{d}y \\ & + O_A\left(\varepsilon^3(1 + \varepsilon|\hat{v}_4|) + \varepsilon(\varepsilon|\hat{v}_4|)^A r^A\right). \end{aligned}$$

Next, we claim that

$$\int_{\mathbf{R}} y^j h(r, y) I(y) \mathrm{d}y = O_A(r^A) + \begin{cases} I(0) & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases} \tag{4.26}$$

To see this, note that $I(y)$ belongs to the class of weight functions considered in [43, Lemma 9]. This settles (4.26) when $j = 0$. When $j > 0$ we truncate the integral to $|y| \leq \sqrt{r}$ and expand $I(y)$ as a Taylor series, before invoking [43, Lemma 8], as in the proof of [43, Lemma 9]. This settles (4.26) when $j > 0$. The statement of the lemma is now obvious. $\square$

Lemma 4.4.5. *Assume that $\varepsilon|\hat{v}_4| > 1$. For each $j \geq 0$, we define*

$$\varphi_j(y) = \Delta^j \psi_y \left((\varepsilon\hat{v}_4)^{-1}\mathbf{a}\right) = \Delta^j \psi_y \left((\varepsilon\hat{c}_4)^{-1}(\hat{c}_1, \hat{c}_2, \hat{c}_3)\right),$$

where ψ_y is given by (4.18). Let $A \geq 0$. Then there exist constants k_j that depend only on j such that

$$\begin{aligned} I_r^*(\boldsymbol{v}) = \; & \frac{\varepsilon\delta(\hat{\boldsymbol{c}})}{(\varepsilon\hat{v}_4)^{\frac{3}{2}}} e\left(-\frac{\|\mathbf{a}\|^2}{2\varepsilon\hat{v}_4}\right) \sum_{j=0}^{A} \frac{k_j}{(\varepsilon\hat{v}_4)^j} \int_{\mathbf{R}} h(r,y) e\left(-\frac{\varepsilon\hat{v}_4 y}{2}\right) \varphi_j(y) \mathrm{d}y \\ & + O_A\left(\frac{\varepsilon^3}{|\varepsilon\hat{v}_4|^{\frac{1}{2}}} + \frac{\varepsilon}{|\varepsilon\hat{v}_4|^{\frac{5}{2}+A}}\right), \end{aligned}$$

where

$$\delta(\hat{\boldsymbol{c}}) = \begin{cases} 1 & \text{if } \varepsilon|\hat{c}_4| \gg |(\hat{c}_1, \hat{c}_2, \hat{c}_3)|, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It will be convenient to set $\lambda = \varepsilon\hat{v}_4$ in the proof of this result, recalling our hypothesis that $|\lambda| > 1$. Our starting point is the expression for $T(y)$ in (4.22), in which $I(y)$ is given by (4.23). By completing the square, we may write

$$T(y) = \frac{\varepsilon}{2}\left(1 + O(|\lambda|\varepsilon^2)\right) e\left(-\frac{\|\mathbf{a}\|^2}{2\lambda}\right) I^*(y),$$

since $|\lambda| > 1$, where

$$I^*(y) = \int_{\mathbf{R}^3} \psi_y\left(\boldsymbol{x} + \frac{\mathbf{a}}{\lambda}\right) e\left(\frac{\lambda}{2}\|\boldsymbol{x}\|^2\right) \mathrm{d}\boldsymbol{x}.$$

If $|\mathbf{a}| \gg \varepsilon|\hat{v}_4|$, then it follows from [43, Lemma 10] that $T(y) \ll_A \varepsilon|\lambda|^{-A}$, for any $A \geq 0$. Alternatively, if $|\mathbf{a}| \ll \varepsilon|\hat{v}_4|$, which is equivalent to $\delta(\hat{\boldsymbol{c}}) = 1$, then all the hypotheses of Lemma 4.4.2 are met. Thus, for any $A \geq 0$, there exist constants k_j

that depend only on j such that

$$I^*(y) = \frac{1}{\lambda^{\frac{3}{2}}} \sum_{j=0}^{A} \frac{k_j \Delta^j \psi_y(\lambda^{-1}\mathbf{a})}{\lambda^j} + O_A\left(\frac{1}{|\lambda|^{\frac{5}{2}+A}}\right).$$

Hence we conclude from (4.22) that

$$T(y) = \frac{\varepsilon\delta(\hat{\boldsymbol{c}})}{2\lambda^{\frac{3}{2}}} e\left(-\frac{\|\mathbf{a}\|^2}{2\lambda}\right) \sum_{j=0}^{A} \frac{k_j \Delta^j \psi_y(\lambda^{-1}\mathbf{a})}{\lambda^j} + O_A\left(\frac{\varepsilon^3}{|\lambda|^{\frac{1}{2}}} + \frac{\varepsilon}{|\lambda|^{\frac{5}{2}+A}}\right).$$

We now wish to substitute this into our expression (4.16) for $I_r^*(\boldsymbol{v})$. In order to control the contribution from the error term, we apply (4.24) with $\ell = 0$. We therefore arrive at the statement of the lemma on redefining k_j to be $k_j/2$. □

It remains to consider the integral

$$\begin{aligned} J_{j,q}(\boldsymbol{c}) &= \int_{\mathbf{R}} h\,(r, y)\, e\left(-\frac{\varepsilon \hat{v}_4 y}{2}\right) \varphi_j(y)\mathrm{d}y \\ &= \int_{\mathbf{R}} h\left(\frac{q}{Q}, y\right) e\left(-\frac{\varepsilon \hat{c}_4 y Q}{2q}\right) \varphi_j(y)\mathrm{d}y, \end{aligned} \tag{4.27}$$

for $j \geq 0$. Recollecting (4.18), all we shall need to know about φ_j is that it is a smooth compactly supported function with bounded derivatives, and that it does not depend on q. (Note that we may assume that $|(\hat{c}_1, \hat{c}_2, \hat{c}_3)| \ll \varepsilon|\hat{c}_4|$ in what follows, since otherwise $\delta(\hat{\boldsymbol{c}}) = 0$.)

Lemma 4.4.6. *Let $\boldsymbol{c} \in \mathcal{C}$ and $k \in \{0, 1\}$. Then*

$$q^k \frac{\partial^k J_{j,q,}(\boldsymbol{c})}{\partial q^k} \ll_j N^\delta.$$

Proof. When $k = 0$ the result follows immediately from (4.24). Suppose next that

$k=1$. Then (4.27) implies that

$$\begin{aligned}
\frac{\partial J_{j,q}(\boldsymbol{c})}{\partial q} &= \frac{1}{Q}\int_{\mathbf{R}} \frac{\partial h\left(r,y\right)}{\partial r} e\left(-\frac{\varepsilon \hat{c}_4 y Q}{2q}\right)\varphi_j(y)\mathrm{d}y \\
&\quad + \int_{\mathbf{R}} \frac{\pi i \varepsilon \hat{c}_4 y Q}{q^2} h\left(r,y\right) e\left(-\frac{\varepsilon \hat{c}_4 y Q}{2q}\right)\varphi_j(y)\mathrm{d}y \\
&= J_1 + J_2,
\end{aligned}$$

say. It follows from (4.24) that $J_1 \ll_j Q^{-1}r^{-1} = q^{-1}$, which is satisfactory. Next, a further application of (4.24) yields

$$J_2 \ll_j \frac{\varepsilon|\hat{c}_4|Q}{q^2}\int_{\mathbf{R}} |yh\left(r,y\right)|\,\mathrm{d}y \ll_j \frac{\varepsilon|\hat{c}_4|Q}{q^2}\cdot r \le \frac{N^\delta}{q},$$

for $\boldsymbol{c} \in \mathcal{C}$. $\square$

4.5 Putting everything together

It is now time to return to (4.2), in order to conclude the proof of Theorem 4.2.1.

4.5.1 The main term

We begin by dealing with the main contribution, which comes from the term $\boldsymbol{c} = \mathbf{0}$. Denoting this by $M(w)$, we see that

$$M(w) = \frac{1}{Q^2}\sum_{q \ll Q} q^{-4} S_q(\mathbf{0}) I_q(\mathbf{0}) + O_A(Q^{-A}), \tag{4.28}$$

for any $A > 0$.

In view of (4.18), $\psi_0(\boldsymbol{x})$ is equal to

$$w_0\left(2\varepsilon^{-2}(-1+\sqrt{1-\varepsilon^2\|\boldsymbol{x}\|^2})\right) w_0\left(\sqrt{\|\boldsymbol{x}\|^2+\varepsilon^{-2}(1-\sqrt{1-\varepsilon^2\|\boldsymbol{x}\|^2})^2}\right).$$

As in (4.19), when $\psi_0(\boldsymbol{x}) \neq 0$ we must have

$$2\varepsilon^{-2}\left(-1+\sqrt{1-\varepsilon^2\|\boldsymbol{x}\|^2}\right) = -\|\boldsymbol{x}\|^2 + O(\varepsilon^2)$$
$$\|\boldsymbol{x}\|^2 + \varepsilon^{-2}(1-\sqrt{1-\varepsilon^2\|\boldsymbol{x}\|^2})^2 = \|\boldsymbol{x}\|^2 + O(\varepsilon^2).$$

In particular it is clear that

$$\sigma_\infty = \int_{\mathbf{R}^3} \psi_0(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \gg 1, \tag{4.29}$$

for an absolute implied constant. We now establish the following result.

Lemma 4.5.1. *We have*

$$I_q(\mathbf{0}) = \tfrac{1}{2}\varepsilon^5 N^2 \sigma_\infty \left(1 + O(\varepsilon^2) + O_A\left((q/Q)^A\right)\right),$$

for any $A > 0$, where σ_∞ is given by (4.29).

Proof. Returning to (4.7), it follows from (4.16) and (4.17) that

$$I_q(\mathbf{0}) = \varepsilon^4 N^2 \int_{\mathbf{R}} h(r,y)K(y)\mathrm{d}y,$$

where

$$K(y) = \int_{\mathbf{R}^3} w_0(\|\boldsymbol{u}\|)w_0(2u_4/\varepsilon)\,\frac{\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3}{2/\varepsilon + 2u_4},$$

and u_4 is given in terms of y, u_1, u_2, u_3 by (4.15). Using (4.20), we may write

$$K(y) = \frac{\varepsilon}{2}\left(1 + O(\varepsilon^2)\right)K^*(y), \quad \text{with } K^*(y) = \int_{\mathbf{R}^3} \psi_y(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

The integral $K^*(y)$ is a smooth weight function belonging to the class of weight functions considered in [43, Lemma 9]. Noting from (4.29) that $K^*(0) = \sigma_\infty$, it therefore follows from this result that

$$\int_{\mathbf{R}} h(r,y)K^*(y)\mathrm{d}y = \sigma_\infty + O_A(r^A),$$

for any $A > 0$. We therefore deduce that

$$I_q(\mathbf{0}) = \tfrac{1}{2}\varepsilon^5 N^2 \sigma_\infty \left(1 + O(\varepsilon^2) + O_A(r^A)\right),$$

which completes the proof of the lemma. $\square$

Now it is clear from §4.3 that $q^{-4}|S_q(\boldsymbol{c})| \le 4q^{-2}|S(m,n;q)|$, for any vector $\boldsymbol{c} \in \mathbf{Z}^4$, where (m,n) is $(N, F(\hat{\boldsymbol{c}})/4)$, $(N/2, F(\hat{\boldsymbol{c}})/2)$ or $(N/4, F(\hat{\boldsymbol{c}}))$ depending on whether $4 \mid q$, $q \equiv 2 \bmod 4$ or $2 \nmid q$, respectively. Hence it follows from (4.5), together with the standard estimate for the divisor function, that

$$\begin{aligned}\sum_{t/2<q\le t} q^{-4}|S_q(\boldsymbol{c})| \ll \sum_{t/2<q\le t} q^{-2}|S(m,n;q)| &\ll_\delta t^{\delta/2} \sum_{t/2<q\le t} \frac{\sqrt{(q,N)}}{q^{3/2}} \\ &\ll_\delta t^{-1/2+\delta/2} N^{\delta/2},\end{aligned} \tag{4.30}$$

for any $t > 1$ and any $\delta > 0$. Returning to (4.28), we may now conclude from Lemma 4.5.1 and (4.30) with $\boldsymbol{c} = \mathbf{0}$, that the contribution to $M(w)$ from $q \le Q^{1-\delta}$ is

$$\begin{aligned}&= \frac{1}{Q^2} \sum_{q \le Q^{1-\delta}} q^{-4} S_q(\mathbf{0}) I_q(\mathbf{0}) + O_A(Q^{-A}) \\ &= \frac{\varepsilon^5 N^2}{2Q^2} \sigma_\infty \mathfrak{S}(Q^{1-\delta}) + O\left(\frac{\varepsilon^7 N^{2+\delta/2}}{Q^2}\right) + O_A(Q^{-A}),\end{aligned}$$

where

$$\mathfrak{S}(t) = \sum_{q \le t} q^{-4} S_q(\mathbf{0}).$$

This sum is absolutely convergent and satisfies $\mathfrak{S}(t) = \mathfrak{S} + O_\delta(t^{-1/2+\delta/2} N^{\delta/2})$, for any $\delta > 0$, by (4.30). Here, in the usual way, $\mathfrak{S}$ is the Hardy–Littlewood product of local densities recorded in (4.4).

Next, on invoking (4.30), once more, the contribution from $q > Q^{1-\delta}$ is

$$\ll_A \frac{\varepsilon^5 N^2}{Q^2} \sum_{q > Q^{1-\delta}} q^{-4}|S_q(\mathbf{0})| + Q^{-A} \ll \frac{\varepsilon^5 N^{2+\delta/2} Q^{\delta/2}}{Q^{5/2}}.$$

Hence we have established the following result, on recalling that $Q = \varepsilon\sqrt{N}$, which shows that the main term is satisfactory for Theorem 4.2.1.

Lemma 4.5.2. *For any $\delta > 0$ we have*

$$M(w) = \frac{\varepsilon^3 N \sigma_\infty \mathfrak{S}}{2} + O_\delta\left(\varepsilon^5 N^{1+\delta} + \varepsilon^{\frac{5}{2}} N^{\frac{3}{4}+\delta}\right).$$

4.5.2 The error term

It remains to analyse the contribution $E(w)$, say, to $\Sigma(w)$ from vectors $\boldsymbol{c} \neq \boldsymbol{0}$ in (4.2). According to our work in §4.3 the value of $S_q(\boldsymbol{c})$ differs according to the residue class of q modulo 4. We have

$$E(w) = \sum_{i \bmod 4} E_i(w),$$

where $E_i(w)$ denotes the contribution from $q \equiv i \bmod 4$. Recall the definition of $\mathcal{C}$ from after the statement of Lemma 4.4.1. In order to unify our treatment of the four cases, we write $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ and we denote by $\mathcal{C}_2$ (resp. $\mathcal{C}_4$) the set of $\boldsymbol{c} \in \mathcal{C}$ for which $2 \nmid c_1 \dots c_4$ (resp. $2 \mid \boldsymbol{c}$). It will also be convenient to set

$$(m_1, n_1) = (m_3, n_3) = (N/4, F(\boldsymbol{c})),$$
$$(m_2, n_2) = (N/2, F(\boldsymbol{c})/2), \quad (m_4, n_4) = (N, F(\boldsymbol{c})/4).$$

In particular, $m_i n_i = NF(\hat{\boldsymbol{c}})/4 > 0$ for $1 \leq i \leq 4$, since $F(\boldsymbol{c}) = F(\hat{\boldsymbol{c}})$.

Let $1 \ll R \ll Q$. We denote by $E_i(w, R)$ the overall contribution to $E_i(w)$ from $q \sim R$. (We write $q \sim R$ to denote $q \in (R/2, R]$.) On recalling (4.7), it follows from our work so far that

$$\begin{aligned} E_i(w,R) &\ll \frac{1}{Q^2} \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ \boldsymbol{c} \neq \boldsymbol{0}}} \left| \sum_{\substack{q \sim R \\ q \equiv i \bmod 4}} q^{-2} S(m_i, n_i; q) I_q(\boldsymbol{c}) \right| \\ &\ll \frac{\varepsilon^4 N^2}{Q^2} \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ \boldsymbol{c} \neq \boldsymbol{0}}} \left| \sum_{\substack{q \sim R \\ q \equiv i \bmod 4}} q^{-2} S(m_i, n_i; q) e_r(-\varepsilon^{-1} \boldsymbol{c}.\boldsymbol{\xi}) I_r^*(\boldsymbol{v}) \right|. \end{aligned} \tag{4.31}$$

Contribution from large q

Suppose first that $R \geq Q^{1-\eta}$, for some small $\eta > 0$. (The choice $\eta = 2\delta$ is satisfactory.) We have

$$e_r(-\varepsilon^{-1}\boldsymbol{c}.\boldsymbol{\xi}) = e\left(\frac{2\sqrt{m_i n_i}}{q}\alpha\right),$$

with

$$|\alpha| = \varepsilon^{-1}|\hat{c}_4| \cdot \frac{Q}{q} \cdot \frac{q}{2\sqrt{m_i n_i}} = \frac{|\hat{c}_4|}{\sqrt{F(\hat{\boldsymbol{c}})}} \leq 1.$$

It now follows from Conjecture 4.1.1 that

$$L(t) = \sum_{\substack{q \leq t \\ q \equiv i \bmod 4}} \frac{S(m_i, n_i; q)}{q} e\left(\frac{2\sqrt{m_i n_i}}{q}\alpha\right) \ll_\delta (tN)^\delta. \tag{4.32}$$

Applying partial summation, based on Lemma 4.4.3, we deduce that

$$\begin{aligned} E_i(w, R) &\ll_\delta \frac{\varepsilon^5 N^{2+O(\delta)}}{Q^3} \cdot \frac{Q^2}{R^2} \cdot \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ \boldsymbol{c} \neq \boldsymbol{0}}} \frac{1}{\max\{1, \varepsilon|\hat{c}_4|Q/R\}^{\frac{3}{2}}} \\ &\ll_\delta \frac{\varepsilon^5 N^{2+O(\delta)}}{QR^2} \cdot \frac{\varepsilon^{-1}R}{Q} \\ &= \frac{\varepsilon^4 N^{2+O(\delta)}}{Q^2 R}. \end{aligned}$$

Since $R \geq Q^{1-\eta}$, we deduce that

$$E_i(w, R) \ll_\delta \frac{\varepsilon^4 N^{2+O(\delta)} Q^\eta}{Q^3} \leq \varepsilon N^{\frac{1}{2}+O(\delta)+\eta}.$$

This is satisfactory for Theorem 4.2.1, on redefining the choice of δ, provided that η is small enough.

Contribution from small q and small $\varepsilon|\hat{v}_4|$

For the rest of the proof we suppose that $R < Q^{1-\eta}$. Let us put

$$\mathbf{b} = (\hat{c}_1, \hat{c}_2, \hat{c}_3),$$

so that $\mathbf{a} = r^{-1}\mathbf{b}$ in Lemmas 4.4.4 and 4.4.5. Let $E_i^{(\text{small})}(w, R)$ denote the contribution to $E_i(w, R)$ from $\boldsymbol{c} \in \mathcal{C}_i$ such that

$$\varepsilon|\hat{c}_4| \leq \frac{R^{1+\delta}}{Q}. \tag{4.33}$$

In this case it is advantageous to apply Lemma 4.4.4 to evaluate $I_r^*(\boldsymbol{v})$. To begin with, we consider the effect of substituting the main term from Lemma 4.4.4. Noting that $(\varepsilon\hat{v}_4)^{-1}\mathbf{a} = (\varepsilon\hat{c}_4)^{-1}\mathbf{b}$ does not depend on q, we deduce from (4.23) that the only dependence on q in $I(y)$ comes through the term

$$e\left(\frac{\varepsilon\hat{v}_4}{2}\|\boldsymbol{x}\|^2 - \mathbf{a}.\boldsymbol{x}\right) = e_r\left(\frac{\varepsilon\hat{c}_4}{2}\|\boldsymbol{x}\|^2 - \mathbf{b}.\boldsymbol{x}\right),$$

in the integrand. Thus, the main term in Lemma 4.4.4 makes the overall contribution

$$\ll \frac{\varepsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ \boldsymbol{c}\neq\mathbf{0} \\ (4.33)\text{ holds}}} \left| \sum_{\substack{q\sim R \\ q\equiv i \bmod 4}} \frac{S(m_i, n_i; q)}{q^2} e_r(-\varepsilon^{-1}\boldsymbol{c}.\boldsymbol{\xi}) I(0) \right| \tag{4.34}$$

to $E_i^{(\text{small})}(w, R)$, where we recall from (4.23) that

$$I(0) = \int_{\mathbf{R}^3} \psi_0(\boldsymbol{x}) e_r\left(\frac{\varepsilon\hat{c}_4}{2}\|\boldsymbol{x}\|^2 - \mathbf{b}.\boldsymbol{x}\right) \mathrm{d}\boldsymbol{x}.$$

If $\boldsymbol{c} \neq \mathbf{0}$ and $|\hat{c}_4| \leq \frac{1}{100}$ then

$$\|\mathbf{b}\|^2 = F(\hat{\boldsymbol{c}}) - \hat{c}_4^2 = F(\boldsymbol{c}) - \hat{c}_4^2 \gg 1.$$

It therefore follows from Lemma 3.4.1 that

$$I(0) \ll_A \left(\frac{q}{|\mathbf{b}|Q}\right)^A \ll_A Q^{-\eta A},$$

since $q \leq Q^{1-\eta}$ in this case. The overall contribution to (4.34) from vectors $\boldsymbol{c}$ such that $|\hat{c}_4| \leq \frac{1}{100}$ is therefore seen to be satisfactory.

On interchanging the sum and the integral we are left with the contribution

$$\ll \frac{\varepsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ |\hat{c}_4|>\frac{1}{100} \\ (4.33) \text{ holds}}} \int_{[-1,1]^3} |M_i(\boldsymbol{x})| \mathrm{d}\boldsymbol{x}, \tag{4.35}$$

where

$$M_i(\boldsymbol{x}) = \sum_{\substack{q\sim R \\ q\equiv i \bmod 4}} \frac{S(m_i, n_i; q)}{q^2} e_r(-\varepsilon^{-1}\boldsymbol{c}.\boldsymbol{\xi}) e_r\left(\frac{\varepsilon\hat{c}_4}{2}\|\boldsymbol{x}\|^2 - \mathbf{b}.\boldsymbol{x}\right).$$

But

$$e_r(-\varepsilon^{-1}\boldsymbol{c}.\boldsymbol{\xi}) e_r\left(\frac{\varepsilon\hat{c}_4}{2}\|\boldsymbol{x}\|^2 - \mathbf{b}.\boldsymbol{x}\right) = e\left(\frac{2\sqrt{m_i n_i}}{q}\alpha\right),$$

with

$$\begin{aligned} \alpha &= \left(-\varepsilon^{-1}\hat{c}_4 + \frac{\varepsilon\hat{c}_4\|\boldsymbol{x}\|^2}{2} - \mathbf{b}.\boldsymbol{x}\right) \cdot \frac{Q}{q} \cdot \frac{q}{2\sqrt{m_i n_i}} \\ &= -\frac{\hat{c}_4}{\sqrt{F(\hat{\boldsymbol{c}})}} + \frac{\varepsilon^2\hat{c}_4\|\boldsymbol{x}\|^2}{2\sqrt{F(\hat{\boldsymbol{c}})}} - \frac{\varepsilon\mathbf{b}.\boldsymbol{x}}{\sqrt{F(\hat{\boldsymbol{c}})}}. \end{aligned}$$

But the inequality $\max\{\|\mathbf{b}\|, |\hat{c}_4|\} \leq \sqrt{F(\hat{\boldsymbol{c}})}$, implies that $|\alpha| \leq 1 + O(\varepsilon)$, since $\boldsymbol{x} \in [-1,1]^3$. Thus it follows from combining partial summation with Conjecture 4.1.1 that $M_i(\boldsymbol{x}) \ll_\delta R^{-1}N^\delta$. (Recall that $\varepsilon^{-1} \leq \sqrt{N}$ and $R \leq Q^{1-\eta} \leq Q$.) Returning to (4.35), we conclude that the overall contribution to $E_i^{(\mathrm{small})}(w, R)$ from the main term in Lemma 4.4.4 is

$$\begin{aligned} \ll_\delta \frac{\varepsilon^5 N^{2+\delta}}{RQ^2} \#\left\{\boldsymbol{c} \in \mathcal{C}_i : |\hat{c}_4| > \tfrac{1}{100} \text{ and (4.33) holds}\right\} &\ll_\delta \frac{\varepsilon^4 N^{2+4\delta} R^\delta}{Q^3} \\ &\ll_\delta \varepsilon N^{\frac{1}{2}+5\delta}. \end{aligned}$$

This is satisfactory for Theorem 4.2.1.

It remains to study the effect of substituting the error term from Lemma 4.4.4 into (4.31). Since $r \leq R/Q \leq Q^{-\eta}$ and $\varepsilon|\hat{v}_4| = r^{-1}\varepsilon|\hat{c}_4| \ll R^{\delta}$, by (4.33), we see that the error term is

$$\begin{aligned}\ll_A \varepsilon^3(1+\varepsilon|\hat{v}_4|) + \varepsilon(1+\varepsilon|\hat{v}_4|)^A r^A &\ll_A \varepsilon^3 R^{\delta} + \varepsilon R^{\delta A} Q^{-\eta A} \\ &\leq \varepsilon^3 R^{\delta} + \varepsilon Q^{A(\delta-\eta)}.\end{aligned}$$

On ensuring that $\delta < \eta$, we see that the second term is an arbitrary negative power of Q and so makes a satisfactory overall contribution to $E_i^{(\text{small})}(w, R)$. In view of (4.30), the contribution from the term $\varepsilon^3 N^{\delta}$ is found to be

$$\ll_{\delta} \frac{\varepsilon^7 N^{2+\delta}}{Q^2 R^{\frac{1}{2}}} \cdot \#\mathcal{C}_i \ll_{\delta} \frac{\varepsilon^7 N^{2+\delta}}{Q^2} \cdot \varepsilon^{-1} N^{4\delta} = \frac{\varepsilon^6 N^{2+5\delta}}{Q^2}, \tag{4.36}$$

since $R \gg 1$. The right hand side is $\varepsilon^4 N^{1+5\delta}$, which is also satisfactory for Theorem 4.2.1, on redefining δ.

Contribution from small q and large $\varepsilon|\hat{v}_4|$

It remains to consider the case $R < Q^{1-\eta}$ and

$$\varepsilon|\hat{c}_4| > \frac{R^{1+\delta}}{Q}. \tag{4.37}$$

Let us write $E_i^{(\text{big})}(w, R)$ for the overall contribution to $E_i(w, R)$ from this final case. Our main tool is now Lemma 4.4.5. Let $A \geq 0$. We begin by considering the effect of substituting the main term from this result into (4.31). This yields the contribution

$$\ll \frac{\varepsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c} \in \mathcal{C}_i \\ (4.37) \text{ holds}}} \delta(\hat{\boldsymbol{c}}) \sum_{j=0}^{A} \frac{|k_j|}{(\varepsilon|\hat{c}_4|Q)^{\frac{3}{2}+j}} |M_{i,j}|, \tag{4.38}$$

where if $J_{j,q}(\boldsymbol{c})$ is given by (4.27), then

$$M_{i,j} = \sum_{\substack{q\sim R\\ q\equiv i \bmod 4}} \frac{S(m_i,n_i;q)}{q} e_r(-\varepsilon^{-1}\boldsymbol{c}.\boldsymbol{\xi}) e_r\left(-\frac{\|\mathbf{b}\|^2}{2\varepsilon\hat{c}_4}\right) q^{\frac{1}{2}+j} J_{j,q}(\boldsymbol{c}).$$

Our plan is to use partial summation to remove the factor $q^{\frac{1}{2}+j}J_{j,q}(\boldsymbol{c})$.

First, as before, we note that

$$e_r(-\varepsilon^{-1}\boldsymbol{c}.\boldsymbol{\xi}) e_r\left(-\frac{\|\mathbf{b}\|^2}{2\varepsilon\hat{c}_4}\right) = e\left(\frac{2\sqrt{m_in_i}}{q}\alpha\right),$$

where

$$\begin{aligned}\alpha &= \left(-\varepsilon^{-1}\hat{c}_4 - \frac{\|\mathbf{b}\|^2}{2\varepsilon\hat{c}_4}\right)\cdot\frac{Q}{q}\cdot\frac{q}{2\sqrt{m_in_i}}\\ &= -\left(\frac{\hat{c}_4}{\sqrt{F(\hat{\boldsymbol{c}})}} + \frac{\|\mathbf{b}\|^2}{2\hat{c}_4\sqrt{F(\hat{\boldsymbol{c}})}}\right).\end{aligned}$$

We have $|\alpha| \leq 1 + O(\varepsilon^2)$, since $\|\mathbf{b}\| \ll \varepsilon|\hat{c}_4|$ when $\delta(\hat{\boldsymbol{c}}) \neq 0$. Applying partial summation, based on (4.32) and Lemma 4.4.6, we deduce that

$$M_{i,j} = O_{j,\delta}(R^{\frac{1}{2}+j}N^{3\delta}).$$

Returning to (4.38), we conclude that the overall contribution to $E_i^{(\mathrm{big})}(w,R)$ from the main term in Lemma 4.4.5 is

$$\ll_{\delta,A} \frac{\varepsilon^5 N^{2+3\delta}}{Q^2}\sum_{j=0}^{A}\sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i\\ (4.37)\text{ holds}}} \frac{R^{\frac{1}{2}+j}}{(\varepsilon|\hat{c}_4|Q)^{\frac{3}{2}+j}} \ll_{\delta,A} \frac{\varepsilon^5 N^{2+3\delta}}{Q^2}\cdot\frac{N^{3\delta}}{\varepsilon Q} = \varepsilon N^{\frac{1}{2}+6\delta}.$$

This is satisfactory for Theorem 4.2.1, on redefining δ.

We must now consider the effect of substituting the error term

$$\ll_A \frac{\varepsilon^3}{|\varepsilon\hat{v}_4|^{\frac{1}{2}}} + \frac{\varepsilon}{|\varepsilon\hat{v}_4|^{\frac{5}{2}+A}}$$

from Lemma 4.4.5 into (4.31). Since $q \sim R$, it follows from (4.37) that $\varepsilon|\hat{v}_4| \gg R^{\delta}$.

The first term is therefore $O(\varepsilon^3)$, which makes a satisfactory overall contribution by (4.36). On the other hand, on invoking once more the argument in (4.30), the second term makes the overall contribution

$$\begin{aligned}
&\ll_A \frac{\varepsilon^5 N^2}{Q^2} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ (4.37) \text{ holds}}} \sum_{q\sim R} \frac{q^{-2}|S(m_i,n_i;q)|}{|\varepsilon \hat{v}_4|^{\frac{5}{2}+A}} \\
&\ll_{A,\delta} \frac{\varepsilon^5 N^{2+\delta}}{R^{\frac{1}{2}}Q^2} \left(\frac{R}{\varepsilon Q}\right)^{\frac{5}{2}+A} \sum_{\substack{\boldsymbol{c}\in\mathcal{C}_i \\ (4.37) \text{ holds}}} \frac{1}{|\hat{c}_4|^{\frac{5}{2}+A}} \\
&\ll_{A,\delta} \frac{\varepsilon^4 N^{2+4\delta} R^{\frac{1}{2}-A\delta}}{Q^3}.
\end{aligned}$$

This is $O_\delta(\varepsilon N^{\frac{1}{2}+4\delta})$ on assuming that A is is chosen so that $A\delta > \frac{1}{2}$. This is also satisfactory for Theorem 4.2.1, which thereby completes its proof.

Chapter 5

Sums of class numbers

5.1 Introduction

The main result of this chapter is the proof of Theorem 1.2.3, which we recall below. Let

$$D(X,l) = \sideset{}{^\flat}\sum_{1\le n\le X} h(-n)h(-n-l),$$

where $\flat$ in the above sum denotes restriction to n such that both $-n$ and $-n-l$ are fundamental discriminants, and such that neither is congruent to $1 \pmod 8$.

Theorem 5.1.1. *Let $l \geq 0$ be an integer, and let $D(X,l)$ be as above. Let*

$$\delta = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a constant $\widehat{\sigma}(l) = \prod_{p\le\infty} \sigma_p(l)$ given in (5.23) such that for all $\varepsilon > 0$ the following asymptotic formula holds,

$$D(X,l) = \frac{\widehat{\sigma}(l)}{576} X^{\frac{3}{2}}(X+l)^{\frac{1}{2}} + O_\varepsilon\left(X^{\frac{3}{2}-\frac{1}{30}}(X+l)^{\frac{1}{2}+\frac{3+\delta}{180}+\varepsilon}\right).$$

Moreover, $\widehat{\sigma}(l) \neq 0$ whenever $\sigma_2(l) \neq 0$, and $\widehat{\sigma}(l) \ll 1$ for an implied constant that is independent of l.

Remark 5.1.2. Theorem 5.1.1 establishes an asymptotic formula for $D(X,l)$ where the main term exceeds the error term as long as $l \ll X^{2-2\varepsilon}$. By contrast, if $a(n)$ are Fourier coefficients of cusp forms of integral weight, the asymptotic formula $\sum_{n\le X} a(n)a(n+l) \ll X^{1-\varepsilon}$ holds whenever $l \ll X^{2-\frac{14}{39}-2\varepsilon}$ (or for $l \ll X^{2-2\varepsilon}$, if one assumes the Ramanujan conjecture, see [3]). The relative strength of our result may be explained by the fact that our problem reduces to a problem involving quadratic forms in six variables, whereas, when $a(n) = d(n)$, one has to deal with a quadratic form in four variables.

In addition to studying shifted sums, as above, one is often interested in studying moments of arithmetic functions, $\sum_{n\le X} a(n)^k$. In the case of $h(-n)$ we have the following asymptotic formula,

$$\frac{1}{X^{k/2}} \sum_{n\le X} h(-n)^k = c(k)X + O(X^{1-\theta}),$$

where the sum ranges over fundamental discriminants. For fixed k, this is due to Wolke [89], who showed that the asymptotic formula holds with $\theta = 1/4$. Lavrik [65] showed that one can take $k \ll \sqrt{\log X}$, and finally, Granville and Soundararajan [38] have shown that the asymptotic formula holds in the wider range $k \ll \log X$. The methods used to prove these results rely on the theory of character sums. Wolke expects the true order of the error term in (5.1) to be $\theta = 1/2$.

We also prove a result analogous to Theorem 5.1.1 for the *non-split* sum.

Theorem 5.1.3. *Let d be a non-negative integer. Set*

$$S(X,d) = \sum_{n\le X}^{\flat} h(-(n^2+d)),$$

where the $\flat$ denotes restriction to fundamental discriminants $-(n^2+d)$ that avoid the congruence class $1 \pmod 8$. *Then there exists a constant $\widetilde{\sigma}(d) = \prod_{p\le\infty} \widetilde{\sigma}_p(d)$ given in* (5.28) *such that for all $\varepsilon > 0$ we have*

$$S(X,d) = \frac{\widetilde{\sigma}(d)}{24} X(X^2+d)^{\frac{1}{2}} + O_\varepsilon(X^{\frac{7}{8}}(X^2+d)^{\frac{37}{72}+\varepsilon}).$$

Moreover, $\widetilde{\sigma}(d) \neq 0$ so long as $\tilde{\sigma}_2(d) \neq 0$.

5.1.1 Correlations involving $r_Q(n)$

Next we will state a more general form of Theorem 1.2.4. Let Q be an integral positive definite quadratic form and let n be an integer. Let $r_Q(n)$ denote the number of representations of n by Q, as in Chapter 1. We establish the following result on correlations between $r_Q(n)$ and $r_Q(n+l)$.

Theorem 5.1.4. *Let Q_1 and Q_2 be two integral positive-definite quadratic forms in $m \geq 3$ variables. Let δ be as in the statement of Theorem 5.1.1. Then there exists a constant $c = c(Q_1, Q_2, l)$ that depends on the quadratic forms Q_i and the shift l, such that for all $\varepsilon > 0$ we have*

$$\sum_{m \leq X} r_{Q_1}(m) r_{Q_2}(m+l) = cX^{\frac{m}{2}}(X+l)^{\frac{m}{2}-1} + O_\varepsilon\left(X^{\frac{m}{2}-\frac{m}{2(2m+1)}}(X+l)^{\frac{m}{2}-1+\frac{3+\delta}{4(2m+1)}+\varepsilon}\right).$$

At the heart of this chapter is Proposition 5.2.1 that counts the number of integer points on $Q_1(\boldsymbol{x}_1) - Q_2(\boldsymbol{x}_2) - l = 0$ satisfying certain congruence conditions - where Q_i are integral, positive definite quadratic forms. The estimates are uniform in l and the congruence conditions, and the proof is modeled on the ideas of [43, Theorem 4]. The main difficulty that we encounter in our analysis is in estimating exponential integrals involving the lopsided weight function in Section 5.2. Ultimately, our error terms are as good as those in [43].

5.2 The main proposition

In this section we adopt the convention that a (k_1+k_2)-tuple $\boldsymbol{x}$ is written $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$, with $\boldsymbol{x}_i$ being k_i-tuples. Let Q_1 and Q_2 be positive-definite integral quadratic forms in k_1 and k_2 variables, respectively, and let $n = k_1 + k_2$. Let A_1, A_2 be positive integers, and $\boldsymbol{a}_i \in (\mathbf{Z}/A_i\mathbf{Z})^{k_i}$ be fixed residue classes. Let $w(\boldsymbol{x})$ be a non-negative smooth function with compact support in $\mathbf{R}^n$ such that $\|w\|_{N,1} \ll_N \|w\|_{1,1}^N$, let l be a non-negative integer and set $Y = X + l$. Define the sum

$$S(\boldsymbol{a}_1, \boldsymbol{a}_2) = \sum_{\substack{\boldsymbol{x} \in \mathbf{Z}^{k_1} : \boldsymbol{x} \equiv \boldsymbol{a}_1 (\mathrm{mod}\ A_1) \\ \boldsymbol{y} \in \mathbf{Z}^{k_2} : \boldsymbol{y} \equiv \boldsymbol{a}_2 (\mathrm{mod}\ A_2) \\ Q_1(\boldsymbol{x}) - Q_2(\boldsymbol{y}) = l}} w\left(\frac{\boldsymbol{x}}{Y^{\frac{1}{2}}}, \frac{\boldsymbol{y}}{X^{\frac{1}{2}}} \right). \tag{5.1}$$

In this section we give an asymptotic formula for $S(\boldsymbol{a}_1, \boldsymbol{a}_2)$,

Proposition 5.2.1. *Let $\varepsilon > 0$ and define*

$$\delta = \begin{cases} 1 & l = 0 \textit{ and } n \textit{ is even,} \\ 0 & \textit{otherwise.} \end{cases}$$

Let

$$N(A_1, A_2, l; p^t) = \# \left\{ \begin{matrix} \boldsymbol{x}_1 \ (\mathrm{mod}\ p^{t+v_p(A_1)}) \\ \boldsymbol{x}_2 \ (\mathrm{mod}\ p^{t+v_p(A_2)}) \end{matrix} : \begin{matrix} \boldsymbol{x}_1 \equiv \boldsymbol{a}_1 \ (\mathrm{mod}\ p^{v_p(A_1)}) \\ \boldsymbol{x}_2 \equiv \boldsymbol{a}_2 \ (\mathrm{mod}\ p^{v_p(A_2)}) \\ p^t \mid Q_1(\boldsymbol{x}_1) - Q_2(\boldsymbol{x}_2) - l \end{matrix} \right\}.$$

Define the local densities

$$c_p(A_1, A_2, l) = \lim_{t \to \infty} \frac{N(A_1, A_2, l; p^t)}{p^{(n-1)t}} \tag{5.2}$$

and

$$c_\infty(w, l) = \lim_{\kappa \to 0} \frac{1}{2\kappa} \int_{|Q_1(\boldsymbol{x}_1) - \frac{X Q_2(x_2) + l}{Q^2}| \le \kappa} w(\boldsymbol{x})\, d\boldsymbol{x}. \tag{5.3}$$

Then with notation as above, we have

$$S(\boldsymbol{a}_1, \boldsymbol{a}_2) = \frac{Y^{\frac{k_1}{2} - 1} X^{\frac{k_2}{2}}}{A_1^{k_1} A_2^{k_2}} \left(c_\infty(w, l) \prod_{p < \infty} c_p(A_1, A_2, l) + O_\varepsilon\left(\|w\|_{1,1}^{\frac{n}{2}} (A_1 A_2)^{\frac{7n}{2}} Y^{\frac{3+\delta}{4} + \varepsilon} X^{-\frac{n}{4}} \right) \right)$$
$$+ O_\varepsilon\left(\|w\|_{1,1}^{n} (A_1 A_2)^{3n} Y^{\frac{n-1+\delta}{4} + \varepsilon} \right),$$

Remark 5.2.2. One can also establish Proposition 5.2.1 by adapting the proof of the main theorem in [43], which uses the classical Hardy-Littlewood circle method. However, it appears difficult to get a result that is uniform in the shift l in a wide range using this method.

5.2.1 Applying the δ-method

Using Theorem 2.1.4 to detect the condition $Q_1(\boldsymbol{x}) - Q_2(\boldsymbol{y}) - l = 0$ in (5.1) with $Q^2 = X + l = Y$ we find that

$$
\begin{aligned}
S(\boldsymbol{a}_1, \boldsymbol{a}_2) &= \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sum_{a \pmod q}^{*} e_q(-al) \sum_{\substack{\boldsymbol{x} \in \mathbf{Z}^{k_1} : \boldsymbol{x} \equiv \boldsymbol{a}_1 \pmod{A_1} \\ \boldsymbol{y} \in \mathbf{Z}^{k_2} : \boldsymbol{y} \equiv \boldsymbol{a}_2 \pmod{A_2}}} e_q(a(Q_1(\boldsymbol{x}) - Q_2(\boldsymbol{y}))\times \\
&\qquad w\left(\frac{\boldsymbol{x}}{Y^{\frac{1}{2}}}, \frac{\boldsymbol{y}}{X^{\frac{1}{2}}}\right) h\left(\frac{q}{Q}, \frac{Q_1(\boldsymbol{x}) - Q_2(\boldsymbol{y}) - l}{Q^2}\right) \\
&= \frac{c_Q}{Q^2} \sum_{q=1}^{\infty} \sum_{a \pmod q}^{*} \sum_{\substack{\boldsymbol{b}_1 \pmod{qA_1} : \boldsymbol{b}_1 \equiv \boldsymbol{a}_1 \pmod{A_1} \\ \boldsymbol{b}_2 \pmod{qA_2} : \boldsymbol{b}_2 \equiv \boldsymbol{a}_2 \pmod{A_2}}} e_q(a(Q_1(\boldsymbol{b}_1) - Q_2(\boldsymbol{b}_2) - l))\times \\
&\qquad \sum_{\substack{\boldsymbol{x} \in \mathbf{Z}^{k_1} : \boldsymbol{x} \equiv \boldsymbol{b}_1 \pmod{qA_1} \\ \boldsymbol{y} \in \mathbf{Z}^{k_2} : \boldsymbol{y} \equiv \boldsymbol{b}_2 \pmod{qA_2}}} w\left(\frac{\boldsymbol{x}}{Y^{\frac{1}{2}}}, \frac{\boldsymbol{y}}{X^{\frac{1}{2}}}\right) h\left(\frac{q}{Q}, \frac{Q_1(\boldsymbol{x}) - Q_2(\boldsymbol{y}) - l}{Q^2}\right).
\end{aligned}
$$

Applying Poisson's summation formula to the sum over $\boldsymbol{x}$ and $\boldsymbol{y}$ we get

$$
S(\boldsymbol{a}_1, \boldsymbol{a}_2) = \frac{c_Q Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{q=1}^{\infty} \frac{1}{q^n} \sum_{\boldsymbol{c} \in \mathbf{Z}^n} S_{q,A_1,A_2}(\boldsymbol{a}, \boldsymbol{c}) I_q(\boldsymbol{c}),
$$

where

$$
\begin{aligned}
S_{q,A_1,A_2}(\boldsymbol{a}, \boldsymbol{c}) &= \sum_{d \pmod q}^{*} \sum_{\substack{\boldsymbol{x}_1 \in (\mathbf{Z}/qA_1\mathbf{Z})^{k_1} : \boldsymbol{x}_1 \equiv \boldsymbol{a}_1 \pmod{A_1} \\ \boldsymbol{x}_2 \in (\mathbf{Z}/qA_2\mathbf{Z})^{k_2} : \boldsymbol{x}_2 \equiv \boldsymbol{a}_2 \pmod{A_2}}} e_q\left(d(Q_1(\boldsymbol{x}_1) - Q_2(\boldsymbol{x}_2) - l) + \boldsymbol{c}.\boldsymbol{x}\right), \\
I_q(\boldsymbol{c}) &= \int_{\mathbf{R}^{k_1} \times \mathbf{R}^{k_2}} w(\boldsymbol{x}) h\left(r, \frac{YQ_1(\boldsymbol{x}_1) - XQ_2(\boldsymbol{x}_2) - l}{Y}\right) e_{qA_1/\sqrt{Y}}(-\boldsymbol{c}_1.\boldsymbol{x}_1)\times \\
&\qquad e_{qA_2/\sqrt{X}}(-\boldsymbol{c}_2.\boldsymbol{x}_2)\, d\boldsymbol{x}.
\end{aligned}
\tag{5.4}
$$

We will see that the main term in the asymptotic formula for $S(\boldsymbol{a}_1, \boldsymbol{a}_2)$ comes from $\boldsymbol{c} = \boldsymbol{0}$, and we now turn to bounding the exponential sums and integrals.

5.2.2 Analysis of the exponential sum

We begin by showing that the exponential sum $S_{q,A_1,A_2}(\boldsymbol{a}, \boldsymbol{c})$ is multiplicative in q.

Lemma 5.2.3. *Let $q = q'q''$, $A_1 = A_1'A_1''$ and $A_2 = A_2'A_2''$ such that $(A_1'q', A_1''q'') = (A_2'q', A_2''q'') = 1$. For $i = 1, 2$ let $\boldsymbol{a}_i'$ and $\boldsymbol{a}_i''$ be given by the congruences $\boldsymbol{a}_i' \equiv \boldsymbol{a}_i \pmod{A_i'}$ and $\boldsymbol{a}_i'' \equiv \boldsymbol{a}_i \pmod{A_i}''$. Then we have*

$$S_{q,A_1,A_2}(\boldsymbol{a}, \boldsymbol{c}) = S_{q',A_1',A_2'}(\boldsymbol{a}_1', \boldsymbol{a}_2', \overline{q''}\boldsymbol{c}) S_{q'',A_1'',A_2''}(\boldsymbol{a}_1'', \boldsymbol{a}_2'', \overline{q'}\boldsymbol{c}),$$

where $\overline{q'}q' \equiv 1 \pmod{q''}$ and $\overline{q''}q'' \equiv 1 \pmod{q'}$.

Proof. The proof follows along standard lines. In the definition of $S_{q,A_1,A_2}(\boldsymbol{a}, \boldsymbol{c})$ in (5.4) write $\boldsymbol{x}_1 = q''A_1''\overline{q''A_1''}\boldsymbol{y}_1 + q'A_1'\overline{q'A_1'}\boldsymbol{z}_1$ and $\boldsymbol{x}_2 = q''A_2''\overline{q''A_2''}\boldsymbol{y}_2 + q'A_2'\overline{q'A_2'}\boldsymbol{z}_2$ where $\boldsymbol{y}_1$ and $\boldsymbol{z}_1$ run modulo $q'A_1'$ and $q''A_1''$ respectively, and $\boldsymbol{y}_2$ and $\boldsymbol{z}_2$ run modulo $q'A_2'$ and $q''A_2''$ respectively. Then we have

$$\begin{aligned}
e_q(aQ_1(\boldsymbol{x}_1) + \boldsymbol{c}_1.\boldsymbol{x}_1) &= e_{q'}(a\overline{q''}Q_1(\boldsymbol{y}_1) + \overline{q''}\boldsymbol{c}_1.\boldsymbol{y}_1) e_{q''}(a\overline{q'}Q_1(\boldsymbol{z}_1) + \overline{q'}\boldsymbol{c}_1.\boldsymbol{z}_1) \\
e_q(-aQ_2(\boldsymbol{x}_2) + \boldsymbol{c}_2.\boldsymbol{x}_2) &= e_{q'}(-a\overline{q''}Q_1(\boldsymbol{y}_2) + \overline{q''}\boldsymbol{c}_2.\boldsymbol{y}_2) e_{q''}(a\overline{q'}Q_2(\boldsymbol{z}_2) + \overline{q'}\boldsymbol{c}_2.\boldsymbol{z}_2).
\end{aligned}$$

Now we write $a = q''u + q'v$, where u runs modulo q' and v runs modulo q''. Observe that $e_q(-al) = e_{q'}(-ul)e_{q''}(-vl)$. This completes the proof. □

With the multiplicativity relation at hand, to ease notation, we write $S_q(\boldsymbol{c}) = S_{q,A_1,A_2}(\boldsymbol{a}, \boldsymbol{c})$. Next, we give a preliminary bound for $S_q(\boldsymbol{c})$, in analogy with [43, Lemma 25].

Lemma 5.2.4. *We have*

$$S_q(\boldsymbol{c}) \ll (A_1, q^2)^{\frac{k_1}{2}} (q, A_2^2)^{\frac{k_2}{2}} q^{1+\frac{n}{2}}.$$

Proof. Let $F(\boldsymbol{x}) = F^0(\boldsymbol{x}) - l = Q_1(\boldsymbol{x}_1) - Q_2(\boldsymbol{x}_2) - l$. By Cauchy's inequality we have

$$|S_q(\boldsymbol{c})| \leq (\varphi(q))^{\frac{1}{2}} \widetilde{S}_q^{\frac{1}{2}}(\boldsymbol{c}),$$

where

$$\widetilde{S}_q(\boldsymbol{c}) = \sum_{d \pmod q}^{*} \left| \sum_{\substack{\boldsymbol{x}_1, \boldsymbol{y}_1 \pmod{A_1 q}: \boldsymbol{x}_1, \boldsymbol{y}_1 \equiv \boldsymbol{a}_1 \pmod{A_1} \\ \boldsymbol{x}_2, \boldsymbol{y}_2 \pmod{A_2 q}: \boldsymbol{x}_2, \boldsymbol{y}_2 \equiv \boldsymbol{a}_2 \pmod{A_2}}} e_q(d(F(\boldsymbol{x}) - F(\boldsymbol{y})) + \boldsymbol{c}.(\boldsymbol{x} - \boldsymbol{y})) \right|.$$

Set $\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{z}$. Then $\boldsymbol{z}_1 \equiv 0 \pmod{A_1}$ and $\boldsymbol{z_2} \equiv 0 \pmod{A_2}$. Furthermore, $F(\boldsymbol{x}) - F(\boldsymbol{y}) = F^0(\boldsymbol{z}) + \nabla F(\boldsymbol{z}).\boldsymbol{y}$.

$$\sum_{\boldsymbol{z}} e_q(dF^0(\boldsymbol{z}) + \boldsymbol{c}.\boldsymbol{z}) \sum_{\substack{\boldsymbol{y}_1 \pmod{A_1 q}: \boldsymbol{y}_1 \equiv \boldsymbol{a}_1 \pmod{A_1} \\ \boldsymbol{y}_2 \pmod{A_2 q}: \boldsymbol{y}_2 \equiv \boldsymbol{a}_2 \pmod{A_2}}} e_q(d\boldsymbol{y}.\nabla F(\boldsymbol{z})).$$

If M is the matrix representing the quadratic form F, then $\nabla F(\boldsymbol{z}) = 2M\boldsymbol{z}$. Moreover,

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

where M_i are the matrices corresponding to the quadratic forms Q_i. The sum over $\boldsymbol{y}$ above is 0 unless $2A_1 M_1 \boldsymbol{z_1} \equiv 0 \pmod q$ and $2A_2 M_2 \boldsymbol{z}_2 \equiv 0 \pmod q$. Since this happens for $O_{Q_1,Q_2}((q, A_1^2)^{k_1}(q, A_2^2)^{k_2})$ of the $\boldsymbol{z}$, we have

$$\widetilde{S}_q(\boldsymbol{c}) \ll (q, A_1^2)^{k_1}(q, A_2^2)^{k_2} q^{1+n}.$$

This completes the proof of the lemma. □

The following is the key result on exponential sums that we shall need, and it is similar to [43, Lemma 28].

Lemma 5.2.5. *Let*

$$\delta = \begin{cases} 1 & l = 0 \text{ and } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\sum_{q\le Z} |S_q(\boldsymbol{c})| \ll (A_1A_2)^n Z^{\frac{3+n+\delta}{2}+\varepsilon} X^{\varepsilon}, \tag{5.5}$$

Proof. When $l=0$ and n is even, the result follows from Lemma 5.2.4. When $l\neq 0$, or n is odd, we factorise $q = u_1u_2v$, into a product of pairwise co-prime integers satisfying the conditions: $(u_1, A_1A_2) = 1$, $u_2 \mid (A_1A_2)^\infty$, u_1u_2 is squarefree and v is squarefull such that $(u_1u_2, v) = 1$. Then we have by Lemma 5.2.3 that

$$S_q(\boldsymbol{c}) = S_{u_1}(\overline{u_2v}\boldsymbol{c})S_{u_2}(\overline{u_1v}\boldsymbol{c})S_v(\overline{u_1u_2}\boldsymbol{c}).$$

$S_{u_1}(\overline{u_2v}\boldsymbol{c})$ will split as a product of Gauss sums to prime moduli, each of which can be computed explicitly. For $S_{u_2}(\overline{u_1v}\boldsymbol{c})$ we will use the trivial bound, and use Lemma 5.2.4 to estimate $S_v(\overline{u_1u_2}\boldsymbol{c})$.

Let M be the matrix that corresponds to the quadratic form $F^0(\boldsymbol{x}) = Q_1(\boldsymbol{x}_1) - Q_2(\boldsymbol{x}_2)$. Let $M^{-1}(\boldsymbol{x})$ denote the quadratic form that corresponds to the matrix M^{-1}, which is well-defined modulo p if $p \nmid \det M$. We use the bounds,

$$\begin{aligned}
S_{u_1}(\overline{u_2v}\boldsymbol{c}) &\ll C^{\omega(u_1)}u_1^{\frac{1+n}{2}}(l, M^{-1}(\boldsymbol{c}), u_1)^{\frac{1}{2}},\\
S_v(\overline{u_1u_2}\boldsymbol{c}) &\ll (v, A_1^2)^{\frac{k_1}{2}}(v, A_2^2)^{\frac{k_2}{2}}v^{1+\frac{n}{2}}, \text{and}\\
S_{u_2}(\overline{u_1v}\boldsymbol{c}) &\ll A_1^{\frac{k_1}{2}}A_2^{\frac{k_2}{2}}(A_1A_2)^{\frac{1}{2}}u_2^{\frac{1+n}{2}}.
\end{aligned}$$

The first bound follows from [43, Lemma 26], where C is a constant that depends only on $\det M$. Moreover, the term $(l, M^{-1}(\boldsymbol{c}), u_1)^{\frac{1}{2}}$ can be omitted if n is odd. The last bound follows from the trivial bound, $S_p(\boldsymbol{c}) \ll p^{1+n}$, and by noticing that if $p \mid u_2$ then $p \mid A_1A_2$. Inserting these bounds into the proof of [43, Lemma 28] we obtain (5.5). □

5.2.3 Estimates for exponential integrals I

Let

$$w_0(x) = \begin{cases} \exp(-(1-x^2)^{-1}), & |x| < 1 \\ 0 & |x| \ge 1 \end{cases} \tag{5.6}$$

be a smooth function with compact support and let $\omega(x) = w_0\left(\frac{x}{6n(\|Q_1\|+\|Q_2\|)}\right)$, where $\|Q\|$ denotes the norm of a quadratic form Q, which is the largest coefficient of Q in absolute value. Let

$$z(\boldsymbol{x}) = \frac{YQ_1(\boldsymbol{x}_1) - XQ_2(\boldsymbol{x}_2) - l}{Y}.$$

Then $\omega(z(\boldsymbol{x})) \gg 1$ whenever $\boldsymbol{x} \in \operatorname{supp}(w)$. We have

$$I_q(\boldsymbol{c}) = \int_{\mathbf{R}^n} w_3(\boldsymbol{x}) f(z(\boldsymbol{x})) e(-\boldsymbol{u}.\boldsymbol{x})\, d\boldsymbol{x},$$

where $f(y) = h(r, y)\omega(y)$, and $w_3(\boldsymbol{x}) = \frac{w(\boldsymbol{x})}{\omega(z(\boldsymbol{x}))}$. Then the function $f(y)$ has compact support. Let $r = q/Q$, then by Lemma 2.1.5 we have the following bound for its Fourier transform,

$$p_r(t) = p(t) = \int_{\mathbf{R}} f(r, y)\omega(y) e(-ty)\, dy \ll_j (r|t|)^{-j}. \tag{5.7}$$

Let $\boldsymbol{u}_1 = \frac{\boldsymbol{c}_1}{qA_1/\sqrt{Y}}$, and $\boldsymbol{u}_2 = \frac{\boldsymbol{c}_2}{qA_2/\sqrt{X}}$. By Fourier inversion we obtain,

$$I_q(\boldsymbol{c}) = \int_{\mathbf{R}} p(t) e(-tl/Y) I(\boldsymbol{u}, t) dt, \tag{5.8}$$

where

$$I(\boldsymbol{u}, t) = \int_{\mathbf{R}^n} w_3(\boldsymbol{x}) e\left(tQ_1(\boldsymbol{x}_1) - t\frac{X}{Y}Q_2(\boldsymbol{x}_2) - \boldsymbol{u}.\boldsymbol{x}\right) d\boldsymbol{x}.$$

The key result in this section is

Lemma 5.2.6. *Let $\varepsilon > 0$ be fixed. Suppose that $\mathbf{c} \neq \mathbf{0}$, and $\|w\|_{N,1} \ll_N \|w\|_{1,1}^N$. Then the following bound holds,*

$$I_q(\boldsymbol{c}) \ll_M r^{-1} \min\left(\frac{|\boldsymbol{c}_1|}{A_1}, \frac{|\boldsymbol{c}_2|\sqrt{X}}{\sqrt{Y}A_2}\right)^{-M}$$

if $\frac{|\boldsymbol{c}_1|}{A_1} \gg \|w\|_{1,1}$ and $\frac{|\boldsymbol{c}_2|\sqrt{X}}{\sqrt{Y}A_2} \gg \|w\|_{1,1}$.

Proof. For $M > 0$, by Lemma 3.4.1 we have

$$I(\boldsymbol{u}, t) \ll_M \|w\|_{M,1} |\boldsymbol{u}|^{-M}, \tag{5.9}$$

when $|t| \ll |\boldsymbol{u}|$. Using (5.7) when $|t| \gg |\boldsymbol{u}|$, we get by (5.8) that

$$I_q(\boldsymbol{c}) \ll \|w\|_{N,1} r^{-1} |\boldsymbol{u}|^{-N} + r^{-N} |\boldsymbol{u}|^{1-N}.$$

If $|\boldsymbol{u}| \gg \|w\|_{1,1}$, the second term dominates the first, and we get

$$I_q(\boldsymbol{c}) \ll \frac{Q}{q} \min\left\{ \left(\frac{|\boldsymbol{c}_1|}{A_1}\right)^{-M}, \left(\frac{|\boldsymbol{c}_2|\sqrt{X}}{\sqrt{Y} A_2}\right)^{-M} \right\}.$$

□

5.2.4 Estimates for exponential integrals II

By Lemma 5.2.6 we have arbitrary polynomial decay for the integral $I_q(\boldsymbol{c})$ unless $|\boldsymbol{u}_1| \leq r^{-1} A_1 \|w\|_{1,1} X^{\varepsilon}$, and $|\boldsymbol{u}_2| \ll r^{-1} A_2 \|w\|_{1,1} X^{\varepsilon}$. To get a finer estimate for the integral in this range, we need the following

Lemma 5.2.7. *Let $\varepsilon > 0$ and $\mathbf{c} \neq \mathbf{0}$. If $\mathbf{c}_1 = \mathbf{0}$. Then we have*

$$I_q(\boldsymbol{c}) \ll (r^{-1}|\boldsymbol{u}_2|)^{\varepsilon} \|w\|_{1,1}^{\frac{n}{2}} r^{-1} |\boldsymbol{u}_2|^{-\frac{n}{2}}. \tag{5.10}$$

Suppose $\mathbf{c}_2 = \mathbf{0}$, and let $\varepsilon > 0$. Then we have

$$I_q(\boldsymbol{c}) \ll (r^{-1}|\boldsymbol{u}_1|)^{\varepsilon} \|w\|_{1,1}^{\frac{n}{2}} r^{-1} |\boldsymbol{u}_1|^{-\frac{n}{2}}. \tag{5.11}$$

If $\boldsymbol{c}_1, \boldsymbol{c}_2 \neq \mathbf{0}$, we have

$$I_q(\boldsymbol{c}) \ll (r^{-1}|\boldsymbol{u}|)^{\varepsilon} \|w\|_{1,1}^{\frac{n}{2}} r^{-1} |\boldsymbol{u}_1|^{-\frac{k_1}{2}} |\boldsymbol{u}_2|^{-\frac{k_2}{2}} \tag{5.12}$$

Proof. We begin by recording the trivial bound, $I_q(\boldsymbol{c}) \ll 1$, which follows from [43, Lemma 15]. Next, using the fact that w is compactly supported, we may write

$$\begin{aligned} I(\boldsymbol{u},t) = \int_{\mathbf{R}^{k_1}} & e(tQ_1(\boldsymbol{x}_1) - \boldsymbol{u}_1.\boldsymbol{x_1})\times \\ & \int_{\mathbf{R}^{k_2}} w_3(\boldsymbol{x}) e\left(-t\frac{X}{Y}Q_2(\boldsymbol{x}_2) - \boldsymbol{u}_2.\boldsymbol{x}_2\right)\, d\boldsymbol{x}_2\, d\boldsymbol{x}_1. \end{aligned}$$

Integrating trivially over $\boldsymbol{x}_1$, we get by (5.9) that

$$I(\boldsymbol{u},t) \ll_N \|w\|_{N,1}|\boldsymbol{u}_2|^{-N}$$

if $|t| \ll \frac{Y}{X}|\boldsymbol{u}_2|$. Arguing similarly with the roles of $\boldsymbol{x}_1$ and $\boldsymbol{x_2}$ interchanged, we also have the bound

$$I(\boldsymbol{u},t) \ll_N \|w\|_{N,1}|\boldsymbol{u}_1|^{-N}$$

if $|t| \ll |\boldsymbol{u}_1|$. Finally, by Lemma 3.4.3 we have,

$$I(\boldsymbol{u},t) \ll_{Q_1,Q_2} \begin{cases} \|w\|_{N,1}|\boldsymbol{u}|^{-N} & |\boldsymbol{u}| \gg |t| \\ \prod_{i=1}^{k_1} \min\left(1, (|t|)^{-\frac{1}{2}}\right) \prod_{j=1}^{k_2} \min\left(1, \left(|t|\frac{X}{Y}\right)^{-\frac{1}{2}}\right) & \forall t \in \mathbf{R}. \end{cases} \tag{5.13}$$

In addition to the dependence on the quadratic forms Q_i, the implied constant for the first bound depends on N, and for the second bound the dependence is also on the L^1 norm of the weight function w. The above bounds are sufficient to prove the lemma.

Since w is assumed to be compactly supported away from the origin, we see that $\|w\|_{1,1} \gg 1$.

Suppose first that $\boldsymbol{u}_1 = \mathbf{0}$. We have by (5.13) and (5.8) that

$$\begin{aligned} I_q(\boldsymbol{c}) &\ll \int_{|t|\ll \frac{Y}{X}|\boldsymbol{u}_2|} |p(t)|\|w\|_{N,1}|\boldsymbol{u}_2|^{-N}\, dt + \int_{|t|\gg \frac{Y}{X}|\boldsymbol{u}_2|} |p(t)| \left(\frac{X}{Y}\right)^{\frac{k_1}{2}} |\boldsymbol{u}_2|^{-\frac{n}{2}}\, dt \\ &\ll r^{-1}\|w\|_{N,1}|\boldsymbol{u}_2|^{-N} + r^{-1}\left(\frac{X}{Y}\right)^{\frac{k_1}{2}} |\boldsymbol{u}_2|^{-\frac{n}{2}}. \end{aligned} \tag{5.14}$$

since $\int_{-\infty}^{\infty} |p(t)|\, dt \ll r^{-1}$. If $|\boldsymbol{u}_2| \gg r^{-2\varepsilon/n}\|w\|_{1,1}$, using the fact that $\|w\|_{N,1} \ll \|w\|_{1,1}^N$, and by choosing N large enough we get that

$$r^{-1}\|w\|_{N,1}|\boldsymbol{u}_2|^{-N} \ll_N r^{-1+N\varepsilon} \ll r^{-1}|\boldsymbol{u}_2|^{-\frac{n}{2}}.$$

If $|\boldsymbol{u}_2| \ll r^{-2\varepsilon/n}\|w\|_{1,1}$, observe that

$$|\boldsymbol{u}_1|^{\frac{n}{2}-\varepsilon} \ll \|w\|_{1,1}^{\frac{n}{2}} r^{-\varepsilon}.$$

As a result, we have that

$$1 \ll (r^{-1}|\boldsymbol{u}_2|)^{\varepsilon}\|w\|_{1,1}^{\frac{n}{2}} r^{-1}|\boldsymbol{u}_2|^{-\frac{n}{2}},$$

and obtain the bound

$$I_q(\boldsymbol{c}) \ll 1 \ll (r^{-1}|\boldsymbol{u}_2|)^{\varepsilon}\|w\|_{1,1}^{\frac{n}{2}} r^{-1}|\boldsymbol{u}_2|^{-\frac{n}{2}},$$

which completes the proof of (5.10). The proof of (5.11) follows from an analogous argument, replacing $\boldsymbol{u}_2$ by $\boldsymbol{u}_1$.

Finally, consider the case when $|\boldsymbol{u}_1|$ and $|\boldsymbol{u}_2|$ are both non-zero. The proof of (5.12) follows by combining (5.10) and (5.11) - observe first that these bounds hold even if $\boldsymbol{c}_1 \neq \mathbf{0}$, $\boldsymbol{c}_2 \neq \mathbf{0}$, respectively. If $|\boldsymbol{u}_1| \ll |\boldsymbol{u}_2|$, we use (5.10) and the fact that $|\boldsymbol{u}_2|^{-\frac{k_1}{2}} \ll |\boldsymbol{u}_1|^{-\frac{k_1}{2}}$. If $|\boldsymbol{u}_2| \ll |\boldsymbol{u}_1|$ we use (5.11), and this completes the proof of the lemma. □

5.2.5 Evaluating $I_q(\mathbf{0})$

Recall that

$$I_q(\mathbf{0}) = \int_{\mathbf{R}^n} w(\boldsymbol{x}) h\left(r, Q_1(\boldsymbol{x}_1) - \frac{XQ_2(\boldsymbol{x}_2) + l}{Y}\right) d\boldsymbol{x}.$$

We show that the following holds,

Lemma 5.2.8. *If $q \ll Q$, we have for all $N \geq 1$ that*

$$I_q(\mathbf{0}) = c_\infty(w,l) + O_N(\|w\|_{N,1} r^N),$$

where

$$c_\infty(w,l) = \lim_{\kappa \to 0} \frac{1}{2\kappa} \int_{|Q_1(\boldsymbol{x}) - \frac{XQ_2(\boldsymbol{y})+l}{Q^2}| \leq \kappa} w(\boldsymbol{x})\, d\boldsymbol{x}.$$

Proof. We follow the proof of [43, Lemma 13], and also keep track of any dependency on w. Let $c_0 = \int_{-\infty}^{\infty} w_0(x)\, dx$, where $w_0(x)$ is defined in (6.14). For $\delta > 0$ define the function

$$w_1(\boldsymbol{x}) = w_\delta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\delta}, \boldsymbol{y}\right) = c_0^{-n} \prod_{i=1}^{n} w_0\left(\frac{x_i - y_i}{\delta}\right) w(\boldsymbol{x}).$$

Then by [43, Lemmas 12,13] and Lemma 2.1.6 we have that

$$\begin{aligned}
I_q(\mathbf{0}) &= \delta^{-n} \int w_1(\boldsymbol{x}) h\left(r, Q_1(\boldsymbol{x}) - \frac{XQ_2(\boldsymbol{y})+l}{Q^2}\right) d\boldsymbol{x}\, d\boldsymbol{y} \\
&= \delta^{-n} \int \left\{ \lim_{\kappa \to 0} \frac{1}{2\kappa} \int_{|Q_1(\boldsymbol{x_1}) - \frac{XQ_2(\boldsymbol{y})+l}{Q^2}| \leq \kappa} w_1(\boldsymbol{x})\, d\boldsymbol{x} + O_N(r^N \|w\|_{N,1}) \right\} d\boldsymbol{y} \\
&= \lim_{\kappa \to 0} \frac{1}{2\kappa} \int_{|Q_1(\boldsymbol{x}) - \frac{XQ_2(\boldsymbol{y})+l}{Q^2}| \leq \kappa} w(\boldsymbol{x})\, d\boldsymbol{x} + O_N(\delta^{-n} r^N \|w\|_{N,1}),
\end{aligned}$$

since $w(\boldsymbol{x}) = \delta^{-n} \int w_\delta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\delta}, \boldsymbol{y}\right) d\boldsymbol{y}$. This completes the proof of the lemma. □

5.2.6 Proof of Proposition 5.2.1

By Lemma 5.2.6, and the fact that $c_Q = 1 + O_A(Q^{-A})$, we get that

$$S(\boldsymbol{a}_1, \boldsymbol{a}_2) = \frac{Y^{k_1/2-1} X^{k_2/2}}{A_1^{k_1} A_2^{k_2}} \sum_{\substack{|\boldsymbol{c_1}| \ll \|w\|_{1,1} A_1 X^\varepsilon \\ |\boldsymbol{c_2}| \ll \|w\|_{1,1} A_2 \frac{\sqrt{Y}}{\sqrt{X}} X^\varepsilon}} \sum_{q \ll Q} \frac{1}{q^n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}) + O_N(Q^{-N}).$$

Define the following subsets of $\mathbf{Z}^n$. Let $\mathcal{C}_1 = \{\mathbf{0}\}$,

$$\mathcal{C}_2 = \left\{ \boldsymbol{c} \in \mathbf{Z}^n : \boldsymbol{c}_1 = \mathbf{0}, 1 \leq |\boldsymbol{c}_2| \ll \|w\|_{1,1} A_2 \frac{\sqrt{Y}}{\sqrt{X}} X^{\varepsilon} \right\}$$
$$\mathcal{C}_3 = \left\{ \boldsymbol{c} \in \mathbf{Z}^n : 1 \leq |\boldsymbol{c}_1| \ll \|w\|_{1,1} A_1 X^{\varepsilon}, \boldsymbol{c}_2 = \mathbf{0} \right\},$$

and

$$\mathcal{C}_4 = \left\{ \boldsymbol{c} \in \mathbf{Z}^n : \begin{array}{c} 1 \leq |\boldsymbol{c}_1| \ll \|w\|_{1,1} A_1 X^{\varepsilon} \\ 1 \leq |\boldsymbol{c}_2| \ll \|w\|_{1,1} A_2 \frac{\sqrt{Y}}{\sqrt{X}} X^{\varepsilon} \end{array} \right\}.$$

We then have

$$\begin{aligned} S(\boldsymbol{a}_1, \boldsymbol{a}_2) &= \frac{Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{i=1}^{4} \sum_{\boldsymbol{c} \in \mathcal{C}_i} \sum_{q \ll Q} \frac{1}{q^n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}) + O_N(Q^{-N}) \\ &= S_1 + S_2 + S_3 + S_4, \end{aligned} \tag{5.15}$$

say.

5.2.6.1 Analysis of the main term

Using the trivial bound, $I_q(\mathbf{0}) \ll 1$ we have by Lemma 5.2.5 that

$$\sum_{q \sim R} \frac{1}{q^n} S_q(\mathbf{0}) I_q(\mathbf{0}) \ll (A_1 A_2)^{2n} R^{\frac{3+\delta}{2}} R^{-\frac{n}{2}}.$$

Hence the terms $q > Q/\|w\|_{1,1} X^{\varepsilon}$ in S_1 make a contribution that is

$$O_{\varepsilon}\left((A_1 A_2)^{2n} \|w\|_{1,1}^{\frac{n}{2}} Y^{\frac{n-1+\delta}{4}+\varepsilon} \right).$$

By Lemma 5.2.8 and [43, Lemma 31] we have

$$S_1 = \frac{c_{\infty}(w,l) Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{q=1}^{\infty} \frac{1}{q^n} S_q(\mathbf{0}) + O\left((A_1 A_2)^{2n} \|w\|_{1,1}^{\frac{n}{2}} Y^{\frac{n-1+\delta}{4}+\varepsilon} \right). \tag{5.16}$$

5.2.6.2 The leading constant

Since $S_q(\boldsymbol{c})$ is multiplicative, it is a standard computation to show that

$$\sum_{q=1}^{\infty} q^{-n} S_q(\mathbf{0}) = \prod_p c_p(A_1, A_2, l),$$

where $c_p(A_1, A_2, l)$ was defined in (5.2).

5.2.6.3 Analysis of the error terms

Recall that

$$S_2 = \frac{Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{\boldsymbol{c} \in \mathcal{C}_2} \sum_{q \ll Q} \frac{1}{q^n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}).$$

By (5.11) we have that

$$I_q(\boldsymbol{c}) \ll \frac{\|w\|_{1,1}^{\frac{n}{2}} A_2^{\frac{n}{2}} Y^{\frac{1}{2}+\varepsilon} q^{\frac{n}{2}-1}}{X^{\frac{n}{4}} |\boldsymbol{c}_2|^{\frac{n}{2}}}.$$

As a result,

$$S_2 \ll \frac{Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}} Y^{\frac{1}{2}+\varepsilon} X^{-\frac{n}{4}} \|w\|_{1,1}^{\frac{n}{2}}}{A_1^{k_1} A_2^{\frac{k_2-k_1}{2}}} \sum_{\boldsymbol{c} \in \mathcal{C}_2} \frac{1}{|\boldsymbol{c}|^{\frac{n}{2}}} \sum_{q \ll \sqrt{Y}} \frac{|S_q(\boldsymbol{c})|}{q^{\frac{n}{2}+1}}$$

By Lemma 5.2.5 we obtain the bound,

$$S_2 \ll (A_1 A_2)^{3n} Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}} Y^{\frac{3+\delta}{4}+\varepsilon} X^{-\frac{n}{4}} \|w\|_{1,1}^{\frac{n}{2}} \sum_{\boldsymbol{c} \in \mathcal{C}_2} \frac{1}{|\boldsymbol{c}|^{\frac{n}{2}}}.$$

To handle the sum over $\boldsymbol{c}$ we use the fact that

$$\sum_{\substack{\boldsymbol{c} \in \mathbf{Z}^d \\ |\boldsymbol{c}| \leq T}} |\boldsymbol{c}|^l \ll \sum_{t \leq T} t^{d+l-1} \ll 1 + T^{d+l}.$$

Hence we get

$$S_2 \ll_\varepsilon \|w\|_{1,1}^{\frac{n}{2}} (A_1A_2)^{3n} Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}} \left(Y^{\frac{3+\delta}{4}+\varepsilon} X^{-\frac{n}{4}} \right) +$$
$$\|w\|_{1,1}^{k_2} (A_1A_2)^{3n} Y^{\frac{n-1+\delta}{4}+\varepsilon}.$$

Next, we consider

$$S_3 = \frac{Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{\boldsymbol{c}\in\mathcal{C}_3} \sum_{q \ll Q} \frac{1}{q^n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}).$$

By (5.11) we have

$$S_3 \ll \frac{Y^{\frac{k_1}{4}-\frac{1}{2}+\varepsilon} X^{\frac{k_2}{2}} Y^{-\frac{k_2}{4}} A_1^{\frac{n}{2}} \|w\|_{1,1}^{\frac{n}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{\boldsymbol{c}\in\mathcal{C}_3} \frac{1}{|\boldsymbol{c}|^{\frac{n}{2}}} \sum_{q \ll \sqrt{Y}} \frac{|S_q(\boldsymbol{c})|}{q^{\frac{n}{2}+1}},$$

and proceeding as before, by Lemma 5.2.5, and summing over $\mathcal{C}_3$ we get,

$$S_3 \ll_\varepsilon \|w\|_{1,1}^{n} (A_1A_2)^{3n} Y^{\frac{n-1+\delta}{4}+\varepsilon}.$$

Finally we have,

$$S_4 = \frac{Y^{\frac{k_1}{2}-1} X^{\frac{k_2}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{\boldsymbol{c}\in\mathcal{C}_4} \sum_{q \ll Q} \frac{1}{q^n} S_q(\boldsymbol{c}) I_q(\boldsymbol{c}).$$

Using (5.12) we see that

$$S_4 \ll \frac{Y^{\frac{k_1}{4}-\frac{1}{2}+\varepsilon} X^{\frac{k_2}{4}} A_1^{\frac{k_1}{2}} A_2^{\frac{k_2}{2}} \|w\|_{1,1}^{\frac{n}{2}}}{A_1^{k_1} A_2^{k_2}} \sum_{\boldsymbol{c}\in\mathcal{C}_4} \frac{1}{|\boldsymbol{c_1}|^{\frac{k_1}{2}} |\boldsymbol{c}_2|^{\frac{k_2}{2}}} \sum_{q \ll \sqrt{Y}} \frac{|S_q(\boldsymbol{c})|}{q^{\frac{n}{2}+1}}.$$

Using Lemma 5.2.5 once again to estimate the sum over q and summing over $\boldsymbol{c} \in \mathcal{C}_4$, we get that

$$S_4 \ll_\varepsilon \|w\|_{1,1}^{n} (A_1A_2)^{3n} Y^{\frac{n-1+\delta}{4}+\varepsilon}.$$

This completes the proof of Proposition 5.2.1.

5.3 Proof of the main theorems

We begin by proving Theorem 5.1.1.

5.3.1 Proof of Theorem 5.1.1

First we show that it is sufficient to work with a smoothed version of $D(X, l)$. Let $1 \le P \le X$ be a parameter that we will choose later, and let α and β be smooth functions with compact support, taking values in $[0, 1]$ satisfying $\alpha^{(j)}(x) \ll_j 1$, and $\beta^{(j)}(x) \ll_j P^j$ such that

$$\alpha(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 1/P \le x \le 1 \\ 0 & \text{if } x \ge 2, \end{cases}$$

and

$$\beta(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } 1/P \le x \le 1 \\ 0 & \text{if } x \ge 1 + 1/P. \end{cases}$$

Define the sum

$$\widetilde{D}(X, l) = \sideset{}{^\flat}\sum_{m-n=l} h(-m)h(-n)\alpha\left(\frac{m}{X+l}\right)\beta\left(\frac{n}{X}\right). \tag{5.17}$$

Then we have

Lemma 5.3.1. *With notation as above, we have for all $\varepsilon > 0$ that*

$$D(X, l) - \widetilde{D}(X, l) \ll X^{3/2}(X + l)^{1/2+\varepsilon}/P.$$

Proof. By the definition of the smooth weights,

$$\begin{aligned}
\widetilde{D}(X,l) &= \sum_{n\le X}^{\flat} h(-n)h(-n-l) \\
&\quad + \sum_{X<n\le X+X/P}^{\flat} h(-n)h(-n-l)\alpha\left(\frac{n+l}{X+l}\right)\beta\left(\frac{n}{X}\right) \\
&\quad + O\Big(\sum_{n<X/P}^{\flat} h(-n)h(-n-l)\Big) \\
&= D(X,l) + O(X^{1/2}(X+l)^{1/2+\varepsilon}X/P).
\end{aligned}$$

□

5.3.2 Reduction to a counting problem

Let $r_3(n)$ be the number of representations of n as a sum of three squares. The key idea is to use an identity of Gauss (see [17, Proposition 5.3.10]),

$$r_3(n) = 12\left(1 - \left(\frac{-n}{2}\right)\right) h(-n), \tag{5.18}$$

which holds when $n < -3$ is a fundamental discriminant. The identity enables us to transform the shifted sum $\sum^{\flat} h(-n)h(-n-l)$ to sums of the form $\sum r_3(n)r_3(n+l)$, which in turn reduces to the problem of counting integer points in bounded regions that lie on the quadratic form $m_1^2+m_2^2+m_3^2-n_1^2-n_2^2-n_3^2-l=0$. This counting problem is executed by appealing to Proposition 5.2.1.

Recall that an integer n is a fundamental discriminant if, $n \equiv 1 \pmod 4$ and square-free, or $n = 4m$ with m square-free and $m \equiv 2$ or $3 \pmod 4$.

To handle the 2-adic congruence conditions, we set up some notation. Let $S = \{1,4\}$. To each $s \in S$ we associate certain residue classes in $\mathbf{Z}/4\mathbf{Z}$, or $\mathbf{Z}/8\mathbf{Z}$. Set $R(1) = \{5\}$, $M(1) = 8$, $R(4) = \{2,3\}$, $M(4) = 4$, and attach weights, $\tau(1,1) = 1, \tau(1,4) = \tau(4,1) = 2$ and $\tau(4,4) = 4$, to pairs $(s,t) \in S \times S$.

Also, for a positive integer A define

$$\varrho(A) = \sum_{\substack{\boldsymbol{x}\in(\mathbf{Z}/A\mathbf{Z})^3 \\ F(\boldsymbol{x})\equiv 0 (\text{mod } A)}} 1.$$

Excluding fundamental discriminants that are congruent to 1 (mod 8) in (5.17), we get by (5.18) that

$$\begin{aligned}
\widetilde{D}(X,l) &= \frac{1}{576} \sum_{(s,t)\in S\times S} \tau(s,t)\times \\
&\qquad \sum_{\substack{sm-tn=l \\ -m\in R(s)(\text{mod } M(s)) \\ -n\in R(t)(\text{mod } M(t))}} \mu^2(m)\mu^2(n)r_3(m)r_3(n)\alpha\left(\frac{sm}{X+l}\right)\beta\left(\frac{tn}{X}\right) \qquad (5.19)\\
&= \sum_{(s,t)\in S\times S} T(s,t),
\end{aligned}$$

say. For the rest of the proof we will use boldface $\boldsymbol{x}$ to denote a 3-tuple (x_1, x_2, x_3), and by $F(\boldsymbol{x}) = |\boldsymbol{x}|_2^2$ we denote the square of the L^2 norm of $\boldsymbol{x}$. We detect the squarefree condition in (5.19) by using the identity $\mu^2(n) = \sum_{d^2|n} \mu(d)$. For instance, we have

$$T(1,1) = \frac{1}{576}\sum_{k=1}^{\infty}\mu(k) \sum_{\substack{m-n=l \\ n\equiv 0(\text{mod } k^2) \\ n\equiv -5(\text{mod } 8) \\ m\equiv -5(\text{mod } 8)}} r_3(n)\mu^2(m)r_3(m)\alpha\left(\frac{m}{X+l}\right)\beta\left(\frac{n}{X}\right).$$

In the following lemma we show that the k-sum can be truncated, and that the tail makes a small contribution. Define

$$w(\boldsymbol{x},\boldsymbol{y}) = \alpha(|\boldsymbol{x}|_2^2)\beta(|\boldsymbol{y}|_2^2). \qquad (5.20)$$

Let $s \in S$. For an integer j define the set

$$\mathcal{A}_j(s) = \left\{ \boldsymbol{a} \in (\mathbf{Z}/M(s)j^2\mathbf{Z})^3 : \begin{array}{l} F(\boldsymbol{a}) \equiv 0 \pmod{j^2} \\ F(\boldsymbol{a}) \in R(s) \ (\text{mod } M(s)) \end{array} \right\}.$$

Lemma 5.3.2. *Fix $\eta > 0$. Then for all $\varepsilon > 0$ we have*

$$\widetilde{D}(X,l) = \frac{1}{576} \sum_{(s,t)\in S\times S} \tau(s,t) \sum_{\substack{j,k\le X^\eta \\ (2,jk)=1,(j,k)^2|l}} \mu(j)\mu(k)\times$$

$$\sum_{\substack{\boldsymbol{a}\in\mathcal{A}_j(s) \\ \boldsymbol{b}\in\mathcal{A}_k(t)}} \sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbf{Z}^3 \\ \boldsymbol{m}\equiv\boldsymbol{a}(\mathrm{mod}\ M(s)j^2) \\ \boldsymbol{n}\equiv\boldsymbol{b}(\mathrm{mod}\ M(t)k^2) \\ sF(\boldsymbol{m})-tF(\boldsymbol{n})=l}} w\left(\frac{\sqrt{s}\boldsymbol{m}}{X+l}, \frac{\sqrt{t}\boldsymbol{n}}{X}\right)$$

$$+ O(X^{3/2-\eta}(X+l)^{1/2+\varepsilon}).$$

The implied constant depends only on ε.

Proof. To simplify notation, we work with $T(1,1)$. The other terms are handled in exactly the same way. Opening up $\mu^2(n)$ we see that

$$T(1,1) = \frac{1}{576}\sum_{k=1}^{\infty}\mu(k) \sum_{\substack{m-n=l \\ n\equiv 0(\mathrm{mod}\ k^2) \\ n\equiv -5(\mathrm{mod}\ 8) \\ m\equiv -5(\mathrm{mod}\ 8)}} r_3(n)\mu^2(m)r_3(m)\alpha\left(\frac{m}{X+l}\right)\beta\left(\frac{n}{X}\right)$$

$$= \sum_{k\le X^\eta} + \sum_{k>X^\eta} = S_1 + S_2, \text{say.}$$

We have

$$S_2 = \frac{1}{576}\sum_{k>X^\eta}\sum_{\boldsymbol{a}_2\in\mathcal{A}_k(1)} \sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbf{Z}^3 \\ F(\boldsymbol{m})\equiv -5(\mathrm{mod}\ 8) \\ \boldsymbol{n}\equiv\boldsymbol{a}_2(\mathrm{mod}\ 8k^2) \\ F(\boldsymbol{m})-F(\boldsymbol{n})=l}} \mu^2(F(\boldsymbol{m}))w\left(\frac{\boldsymbol{m}}{X+l}, \frac{\boldsymbol{n}}{X}\right)$$

By choice of our weight function, $|\boldsymbol{m}| \ll (X+l)^{\frac{1}{2}}$ and $|\boldsymbol{n}| \ll X^{\frac{1}{2}}$. Furthermore, for each fixed $\boldsymbol{n}$, the number of $\boldsymbol{m}$ such that $F(\boldsymbol{m}) - F(\boldsymbol{n}) - l = 0$ is $O\left((X+l)^{\frac{1}{2}+\varepsilon}\right)$. Since $k \ll X^{\frac{1}{2}}$ we have,

$$S_2 \ll (X+l)^{\frac{1}{2}} \sum_{k>X^\eta}\sum_{\boldsymbol{a}_2\in\mathcal{A}_k(1)} \sum_{\boldsymbol{n}\equiv\boldsymbol{a}_2 \pmod{8k^2}} 1$$

$$\ll (X+l)^{\frac{1}{2}} \sum_{k>X^\eta}\sum_{k^2|n} r_3(n) \ll X^{3/2-\eta}(X+l)^{1/2+\varepsilon}.$$

We repeat this process by opening up $\mu^2(m)$ in S_1 to complete the proof. $\square$

Lemma 5.3.3. *Let $(s,t) \in S \times S$. Define the sum*

$$T_q(j,k,(s,t);l) = \sum_{\substack{\boldsymbol{a}\in\mathcal{A}_j(s)\\ \boldsymbol{b}\in\mathcal{A}_k(t)}} \sum_{\substack{(d,q)=1\\ \boldsymbol{x}(\bmod M(s)qj^2)\\ \boldsymbol{x}\equiv\boldsymbol{a}(\bmod M(s)j^2)\\ \boldsymbol{y}(\bmod M(t)qk^2)\\ \boldsymbol{y}\equiv\boldsymbol{b}(\bmod M(t)k^2)}} e_q\left(d(F(\boldsymbol{x}) - F(\boldsymbol{y}) - l)\right).$$

Let $(2,jk)=1$. For p a prime, let $j_p = v_p(j)$ and $k_p = v_p(k)$ be the p-adic valuations of j and k respectively. Let

$$N(j,k,(s,t);2^\alpha) = \#\left\{\begin{matrix}\boldsymbol{x} \pmod{2^\alpha}\\ \boldsymbol{y} \pmod{2^\alpha}\end{matrix} : \begin{matrix} 2^\alpha \mid sF(\boldsymbol{x}) - tF(\boldsymbol{y}) - l \\ F(\boldsymbol{x}) \in R(s) \pmod{M(s)} \\ F(\boldsymbol{y}) \in R(t) \pmod{M(t)}\end{matrix}\right\}$$

and

$$N(j,k;p^\alpha) = \#\left\{\begin{matrix}\boldsymbol{x} \pmod{p^\alpha}\\ \boldsymbol{y} \pmod{p^\alpha}\end{matrix} : \begin{matrix} p^\alpha \mid sF(\boldsymbol{x}) - tF(\boldsymbol{y}) - l \\ p^{2j_p} \mid F(\boldsymbol{x}), p^{2k_p} \mid F(\boldsymbol{y})\end{matrix}\right\}$$

for $p \neq 2$. Define the local densities

$$\gamma_2((s,t);l) = \lim_{\alpha\to\infty} \frac{N(j,k,(s,t);2^\alpha)}{2^{5\alpha}}$$

and

$$\gamma_p(j,k;l) = \lim_{\alpha\to\infty} \frac{N(j,k;p^\alpha)}{p^{5\alpha}}.$$

Then for all $\varepsilon > 0$ we have

$$\sum_{q\le Z} \frac{T_q(j,k,(s,t);l)}{q^6} = \gamma(j,k,(s,t);l) + O_\varepsilon(\varrho(j^2)\varrho(k^2)(jk)^{24} Z^{-\frac{3-\delta}{2}+\varepsilon}),$$

where $\gamma(j,k,(s,t);l) = (M(s)j^2)^3(M(t)k^2)^3\gamma_2(j,k,(s,t);l)\prod_{2<p<\infty}\gamma_p(j,k;l)$.

Proof. To prove the lemma, we make the following claim, which is immediate from Lemma 5.2.3.

Claim: If $q = q_1q_2$, $j = j_1j_2$ and $k = k_1k_2$ with $(M(s)q_1j_1, q_2j_2) = 1$ and $(M(t)q_1k_1, q_2k_2) = 1$, then

$$T_q(j,k,(s,t);l) = T_{q_1}(j_1,k_1,(s,t);l)T_{q_2}(j_2,k_2,(s,t);l).$$

As a result, if $q = \prod p^{q_p}$, $j = \prod p^{j_p}$ and $k = \prod p^{k_p}$, then we have

$$T_q(j,k;l) = \prod_p T_{p^{q_p}}(p^{j_p}, p^{k_p}, (s,t);l).$$

By Lemma 5.2.5 we get that

$$\sum_{q\le Z} \frac{T_q(j,k,(s,t);l)}{q^6} = \sum_{q=1}^{\infty} \frac{T_q(j,k,(s,t);l)}{q^6} + O(\varrho(j^2)\varrho(k^2)(jk)^{24}Z^{-\frac{3-\delta}{2}+\varepsilon}).$$

Therefore,

$$\sum_{q\le Z} \frac{T_q(j,k,(s,t);l)}{q^6} = \prod_p \sum_{k=0}^{\infty} \frac{T_{p^k}(p^{j_p}, p^{k_p}, (s,t);l)}{p^{6k}} + O(\varrho(j^2)\varrho(k^2)(jk)^{24}Z^{-\frac{3-\delta}{2}+\varepsilon}).$$

By a standard argument (e.g. see [46, Lemma 2.2]) it follows that

$$\sum_{q\le Z} \frac{T_q(j,k,(s,t);l)}{q^6} = \gamma(j,k,(s,t);l) + O(\varrho(j^2)\varrho(k^2)(jk)^{24}Z^{-\frac{3-\delta}{2}+\varepsilon}).$$

This completes the proof of the lemma. □

5.3.2.1 Applying the main proposition

We apply Proposition 5.2.1 and Lemma 5.2.8 to each of the terms that appear in Lemma 5.3.2, with $Q_1(\boldsymbol{x}) = Q_2(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2$, and $A_1 = M(s)j^2, A_2 = M(t)k^2$ and $Y = X + l$. Observe that our weight function in (5.20) satisfies $\|w\|_{N,1} \ll_N P^N \asymp \|w\|_{1,1}^N$. Moreover, from the nature of the function w, we can take $\delta = P^{-\frac{1}{2}}$ in Lemma 5.2.8. Putting everything together, and using the fact that $\varrho(j^2) \le j^6$, we get that

$$\widetilde{D}(X,l) = \frac{X^{\frac{3}{2}}Y^{\frac{1}{2}}}{576} \sum_{(s,t)\in S\times S} \tau(s,t)\sigma_\infty(w,(s,t),l)\times$$
$$\sum_{\substack{j,k\le X^\eta \\ (2,jk)=1 \\ (j,k)^2|l}} \frac{\mu(j)\mu(k)\gamma(j,k,(s,t);l)}{M(s)^3j^6M(t)^3k^6} +$$
$$O_\varepsilon\left(X^{\frac{3}{2}+38\eta}Y^{\frac{-1+\delta}{4}+\varepsilon} + P^6X^{38\eta}Y^{\frac{5+\delta}{4}+\varepsilon} + X^{\frac{3}{2}-\eta}Y^{\frac{1}{2}+\varepsilon}\right),$$

where

$$\sigma_\infty(w,(s,t),l) = \lim_{\kappa\to 0}\frac{1}{2\kappa}\int_{|sF(\boldsymbol{x})-\frac{tXF(\boldsymbol{y})+l}{Q^2}|\le\kappa} w((\sqrt{s}\boldsymbol{x}_1,\sqrt{t}\boldsymbol{x}_2))\,d\boldsymbol{x}$$
$$= \frac{\sigma_\infty(w,(1,1),l)}{\tau^3(s,t)}$$

is the singular integral that corresponds to $T(s,t)$ in Lemma 5.2.8. The first error term above comes from applying Lemma 5.3.3, the second from the application of Proposition 5.2.1, and the last error term results from invoking Lemma 5.3.2. It is easy to see that $\frac{\gamma(j,k,(s,t);l)}{M(s)^3j^6M(t)^3k^6} \ll (jk)^{-\frac{3}{2}}$, so we may extend the j and k sums to ∞ to get

$$\widetilde{D}(X,l) = \frac{\widehat{\sigma}_w(l)}{576}X^{\frac{3}{2}}Y^{\frac{1}{2}} + O\left(X^{\frac{3}{2}+38\eta}Y^{\frac{-1+\delta}{4}+\varepsilon} + P^6X^{38\eta}Y^{\frac{5+\delta}{4}+\varepsilon} + X^{\frac{3}{2}-\eta}Y^{\frac{1}{2}+\varepsilon}\right), \tag{5.21}$$

where

$$\widehat{\sigma}_w(l) = \sigma_\infty(w,(1,1),l)\prod_{p<\infty}\sigma_p(l), \tag{5.22}$$

$$\sigma_p(l) = \begin{cases} \sum_{(s,t)\in S\times S}\frac{\gamma_2((s,t),l)}{\tau(s,t)^2} & p=2 \\ \gamma_p(1,1;l)-\gamma_p(p,1;l)-\gamma_p(1,p;l)+\gamma_p(p,p;l) & 2<p<\infty. \end{cases}$$

5.3.2.2 Removing the weight w

By our choice of test function w it follows that

$$\sigma_\infty(w,(1,1),l) = \sigma_\infty(l) + O(1/P),$$

$\sigma_\infty(l) = \lim_{\kappa\to 0}\frac{1}{2\kappa}\int_{|F(\boldsymbol{x}_1)-\frac{XF(\boldsymbol{x}_2)+l}{X+l}|\le\kappa} d\boldsymbol{x}$ and the integral is over the region

$$\mathcal{R} = \left\{\boldsymbol{x}\in\mathbf{R}^6 : |F(\boldsymbol{x}_1)|\le 1, |F(\boldsymbol{x}_2)|\le 1\right\}.$$

Therefore, by taking $X^\eta = P = X^{\frac{1}{30}}Y^{-\frac{3+\delta}{180}-2\varepsilon}$, it follows from Lemma 5.3.1 and (5.21) that

$$D(X,l) = \frac{\widehat{\sigma}(l)}{576}X^{\frac{3}{2}}Y^{\frac{1}{2}} + O_\varepsilon\left(X^{\frac{3}{2}-\frac{1}{30}}Y^{\frac{1}{2}+\frac{3+\delta}{180}+\varepsilon}\right),$$

where

$$\widehat{\sigma}(l) = \prod_{p\le\infty}\sigma_p(l). \tag{5.23}$$

This completes the proof of Theorem 5.1.1.

Remark 5.3.4. It is easy to explicitly compute the singular integral. Indeed, we have for $l\neq 0$ that

$$\sigma_\infty(l) = \frac{\pi^2}{3X^{\frac{3}{2}}(X+l)^{\frac{1}{2}}}\left\{(2X+l)\sqrt{X(X+l)} - l^2\operatorname{arcsinh}\left(\sqrt{\frac{X}{l}}\right)\right\}. \tag{5.24}$$

To see this, recall that

$$\sigma_\infty(l) = \lim_{\kappa\to 0}\frac{1}{2\kappa}\int_{|F(\boldsymbol{x}_1)-\frac{XF(\boldsymbol{x}_2)+l}{X+l}|\le\kappa} d\boldsymbol{x}.$$

Integrating first over $\boldsymbol{x}_1$ we have that

$$\begin{aligned}\sigma_\infty(l) &= \lim_{\kappa\to 0}\frac{1}{2\kappa}\frac{4\pi}{3}\int\left(\frac{XF(\boldsymbol{x}_2)+l}{X+l}+\kappa\right)^{\frac{3}{2}} - \left(\frac{XF(\boldsymbol{x}_2)+l}{X+l}-\kappa\right)^{\frac{3}{2}} d\boldsymbol{x}_2\\ &= \frac{4\pi}{3}\frac{1}{\sqrt{X+l}}\int_{F(\boldsymbol{x}_2)\le 1}\sqrt{XF(\boldsymbol{x}_2)+l}\,d\boldsymbol{x}_2\end{aligned}$$

Switching to spherical co-ordinates, we find that

$$\sigma_\infty(l) = \frac{16\pi^2}{3\sqrt{X+l}} \int_0^1 r^2\sqrt{r^2X+l}\,dr, \tag{5.25}$$

and (5.24) follows. As a result, we see that $\sigma_\infty(l) = \frac{4\pi^2}{3} + O(X^{-\varepsilon})$ whenever $l \ll X^{1-2\varepsilon}$. Denote by I the integral over r in (5.25). Set $r^2X = t$. Then we have that

$$\begin{aligned} I &= \frac{1}{2X^{\frac{3}{2}}} \int_0^X t^{\frac{1}{2}}(t+l)^{\frac{1}{2}}\,dt \\ &= \frac{\sqrt{X+l}}{3} - \frac{1}{6X^{\frac{3}{2}}} \int_0^X t^{\frac{3}{2}}(t+l)^{-\frac{1}{2}}\,dt. \end{aligned}$$

As a result, when $l \gg X^{1+2\varepsilon}$ we find that $\sigma_\infty(l) = \frac{16\pi^2}{9} + O(X^{-\varepsilon})$. Moreover, in the range $0 \le l \ll X^{2-2\varepsilon}$ we have that $1 \ll \sigma_\infty(l) \ll 1$, and the implied constants are absolute.

5.3.3 Proof of Theorem 5.1.3

The proof of Theorem 5.1.3 is similar to the proof of Theorem 5.1.1, so we will only give a brief outline. Here we adopt the notation that a 4-tuple $\boldsymbol{x}$ is written $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$, and $\boldsymbol{x}_1$ is a 3-tuple. Once again it suffices to consider the following weighted analogue of $S(X,d)$,

$$\tilde{S}(X,d) = \sum\nolimits^{\flat} \beta(n/X)h(-(n^2+d)). \tag{5.26}$$

Let $Q_1(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2$ and $Q_2(x) = x^2$.

As before, we need some notation to handle the 2-adic congruence conditions. Let $S = \{3,4,8\}$. Let $M(4) = M(8) = 16$ and $M(3) = 8$. Let $\tau(4) = \tau(8) = 2$ and $\tau(3) = 1$. For $s \in S$ define

$$\mathcal{A}_j(s) = \left\{ \boldsymbol{a}_1 \in \left(\mathbf{Z}/M(s)j^2\mathbf{Z}\right)^3 : \begin{array}{l} Q_1(\boldsymbol{a}_1) \equiv s \pmod{M(s)} \\ Q_1(\boldsymbol{a}_1) \equiv 0 \pmod{j^2} \end{array} \right\}.$$

Since we are excluding fundamental discriminants $-(n^2+d)$ that are congruent to 1 (mod 8) we get, for $\eta > 0$ and any $\varepsilon > 0$ that

$$\widetilde{S}(X,d) = \frac{1}{24}\sum_{s\in S}\tau(s)\sum_{\substack{j\le X^\eta\\(2,j)=1}}\mu(j)\sum_{\boldsymbol{a}_1\in\mathcal{A}_j(s)}\sum_{\substack{\boldsymbol{x}_1\equiv\boldsymbol{a}_1 \,(\mathrm{mod}\ M(s)j^2)\\ Q_1(\boldsymbol{x}_1)-Q_2(\boldsymbol{x}_2)=d}}\alpha\left(\frac{Q_1(\boldsymbol{x}_1)}{X^2+d}\right)\beta\left(\frac{\boldsymbol{x}_2}{X}\right)$$
$$+ O_\varepsilon(X^{1-\eta}(X^2+d)^{\frac{1}{2}+\varepsilon}).$$

Let

$$N(s,d;2^t) = \#\left\{\boldsymbol{x} \pmod{2^t} \quad : \begin{matrix} 2^t \mid Q_1(\boldsymbol{x}_1)-Q_2(\boldsymbol{x}_2)-d \\ Q_1(\boldsymbol{x}_1)\equiv s \pmod{M(s)} \end{matrix}\right\},$$

$$N(j,d;p^t) = \#\left\{\boldsymbol{x} \pmod{p^t} \quad : \begin{matrix} p^t \mid Q_1(\boldsymbol{x}_1)-Q_2(\boldsymbol{x}_2)-d \\ p^{2v_p(j)} \mid Q_1(\boldsymbol{x}_1) \end{matrix}\right\}$$

for $p \neq 2$. Define the local densities

$$\gamma_2(s,d) = \lim_{t\to\infty}\frac{N(s,d;2^t)}{2^{3t}},$$

$$\gamma_p(j;d) = \lim_{t\to\infty}\frac{N(j,d;p^t)}{p^{3t}}$$

and

$$\gamma_\infty(d) = \lim_{\kappa\to 0}\frac{1}{2\kappa}\int_{|Q_1(\boldsymbol{x}_1)-\frac{X^2Q_2(\boldsymbol{x}_2)+d}{X^2+d}|\le\kappa} d\boldsymbol{x},$$

where the integral on the right is over the region

$$\mathcal{R} = \left\{\boldsymbol{x}\in\mathbf{R}^4 : |Q_1(\boldsymbol{x}_1)|\le 1, |Q_2(\boldsymbol{x}_2)|\le 1\right\}.$$

Following the proof of Theorem 5.1.1, and replacing X by X^2, and Y by (X^2+d) we get that

$$\begin{aligned}\widetilde{S}(X,d) = \frac{\widetilde{\sigma}(d)}{24}&X(X^2+d)^{\frac{1}{2}} + \\ &O_\varepsilon\left(P^4X^{13\eta}(X^2+d)^{\frac{3}{4}+\varepsilon}+X^{1-\eta}(X^2+d)^{\frac{1}{2}+\varepsilon}\right),\end{aligned} \tag{5.27}$$

where $\widetilde{\sigma}(d) = \prod_{p\le\infty} \widetilde{\sigma}_p(d)$ and

$$\widetilde{\sigma}_p(d) = \begin{cases} \sum_{s\in S} \tau(s)\gamma_2(s,d) & p = 2 \\ \gamma_p(1;d) - \gamma_p(p;d) & 2 < p < \infty \\ \gamma_\infty(d) & p = \infty. \end{cases} \tag{5.28}$$

To complete the proof, take $X^\eta = P = X^{\frac{1}{18}}(X^2+d)^{-\frac{1}{72}-2\varepsilon}$ to get the desired estimate for $S(X,d)$, since $S(X,d) - \tilde{S}(X,d) \ll_\varepsilon X(X^2+d)^{\frac{1}{2}+\varepsilon}/P$.

5.3.4 Proof of Theorems 1.2.4 and 5.1.4

Let $Q(x_1,x_2) = x_1^2 + x_2^2$. To prove Theorem 1.2.4, we start with the smoothed sum $S = \sum_{m-n=l} \alpha\left(\frac{m}{X+l}\right)\beta\left(\frac{n}{X}\right) r(m)r(n)$, and we see that it differs from the unsmoothed sum by at most $O(X^{1+\varepsilon}/P)$. Applying Proposition 5.2.1 with $Q_1 = Q_2 = Q$, $A_1 = A_2 = 1$ and $w(\boldsymbol{x}) = \alpha(Q_1(\boldsymbol{x}_1))\beta(Q_2(\boldsymbol{x}_2)$, we get

$$S = c'(l)X + O_\varepsilon(P^4(X+l)^{\frac{3}{4}+\varepsilon}),$$

where $c'(l) = c_\infty(w,l)\prod_p c_p(l)$, with $c_\infty(w,l)$ and $c_p(l)$ are as in (5.3) and (5.2). As before, we have

$$c_\infty(w,l) = \lim_{\kappa\to 0} \frac{1}{2\kappa} \int_{|Q(\boldsymbol{x}_1) - \frac{XQ(\boldsymbol{x}_2)+l}{X+l}| \le \kappa} d\boldsymbol{x} + O(1/P),$$

where we integrate over the region

$$\mathcal{R} = \left\{\boldsymbol{x} \in \mathbf{R}^4 : |Q(\boldsymbol{x}_1)| \le 1, |Q(\boldsymbol{x}_2)| \le 1\right\}.$$

Since we are integrating over discs in $\mathbf{R}^2$, it is easy to see that $c_\infty(w,l) = \pi^2 + O(1/P)$. Setting $c(l) = \pi^2 \prod_{p<\infty} c_p(l)$ and $P = X^{\frac{1}{5}}(X+l)^{-\frac{3}{20}-2\varepsilon}$ we get that

$$\sum_{n\le X} r(n)r(n+l) = c(l)X + O_\varepsilon\left(X^{\frac{4}{5}}(X+l)^{\frac{3}{20}+\varepsilon}\right).$$

It is well-known that $c(l) \ll l^{\varepsilon}$, and $c(l) \neq 0$ if and only if $c_p(l) \neq 0$, if and only if the equation $x_1^2 + x_2^2 - x_3^2 - x_4^2 - l = 0$ has a solution in $\mathbf{Z}_p$.

The proof of Theorem 5.1.4 is similar, and follows at once from Proposition 5.2.1 by taking $A_1 = A_2 = 1$, and by setting $w(\boldsymbol{x}) = \alpha(Q_1(\boldsymbol{x}_1))\beta(Q_2(\boldsymbol{x}_2))$, with $P = X^{\frac{m}{2(2m+1)}}(X+l)^{-\frac{3+\delta}{4(2m+1)}}$.

Chapter 6

Sums of Hecke eigenvalues over thin sequences

This chapter is devoted to the proof of Theorem 1.2.5, and we begin by recalling the basic setup. Let $\lambda(n)$ be the normalised Fourier coefficients of a holomorphic Hecke cusp form of full level and weight k. Let $w(\boldsymbol{x}) \in C_0^\infty(\mathbf{R}^4)$ be a smooth function with support in $[1/2, 2]^4$, and let $F(\boldsymbol{x})$ be a non-singular diagonal quadratic form in 4 variables, which we will henceforth fix to be

$$F(\boldsymbol{x}) = A_1x_1^2 + \ldots + A_4x_4^2,$$

for non-zero integers $A_1, \ldots, A_4$. Our main object of interest will be

$$N(\lambda; X) = \sum_{\boldsymbol{x}\in\mathbf{Z}^4 : F(\boldsymbol{x})=0} \lambda(x_1)w(X^{-1}\boldsymbol{x}).$$

6.1 Introduction

The study of averages of arithmetic functions along thin sequences is a central topic in analytic number theory. For instance, the sum $\sum_{n\leq X} a(p(n))$, where $p(n) = n^2+bn+c$ is an integer polynomial, and $a(n)$ are Fourier coefficients of automorphic forms, has been widely studied. For this sum, Hooley [49] established an asymptotic formula with a power-saving error term when $a(n) = d(n)$, and $p(n)$ is irreducible. The case where $a(n)$ are Fourier coefficients of cusp forms was first settled by Blomer [4], and later refined by Templier and Tsimerman [85]. However, the analogous sum over the primes, i.e. the sum $\sum_{n \le X} \Lambda(p(n))$, where $\Lambda(n)$ is the von Mangoldt function, is much harder to estimate, and this is a long standing open problem.

Mean values of arithmetic functions over polynomials of higher degree are poorly understood; obtaining an asymptotic formula for the sum $\sum_{n \le X} d(n^3+2)$ would represent a significant breakthrough in the subject. However, in the case of polynomials in more than variable, several results have been established. Among the most striking results in this regime are by Friedlander and Iwaniec [34] on the existence of infinitely many primes of the form $x^2 + y^4$, and by Heath-Brown [44] on primes of the form $x^3 + 2y^3$.

Analogously, for the divisor function, sums of the form $\sum_{m,n \le X} d(|B(m,n)|)$, where $B(u,v)$ is an integral binary form of degree 3 or 4, have been investigated by several authors. For irreducible binary cubic forms, Greaves [39] gave an asymptotic formula for the aforementioned sum, and the sum over irreducible quartic forms was handled by Daniel [20]. The case when $B(m,n)$ is not irreducible has also been considered; for example, such sums have been of much interest in problems relating to Manin's conjecture for del Pezzo surfaces. See [9], where cubic forms are considered, and [21], [22], [23] and [45] that treat the case of quartic forms. All such approaches were directly inspired by the argument developed by Daniel [20].

Continuing in the same vein as the aforementioned results is the following theorem, which follows from Theorem 1.2.5. In principle, our result corresponds to the case where a cubic form $B(m,n)$ splits over $\mathbf{Q}$ as the product of a linear and a quadratic form.

Theorem 6.1.1. *Let $\lambda(n)$ be the normalised Fourier coefficients of a holomorphic Hecke cusp form f of full level and weight k, and let $r(n)$ be the number of representations of an integer as a sum of two squares. Let A and B be non-zero integers.*

Then there exists $\delta > 0$, independent of A and B, such that

$$\sum_{m,n \leq X} r(Am^2 + Bn^2)\lambda(m) \ll_{f,A,B} X^{2-\delta}.$$

Although we have stated this with the r-function, our methods could potentially be adapted to deal with the divisor function. It is worth emphasising that existing results on divisor sums over binary cubic and quartic forms have largely relied on arguments involving the geometry of numbers; one cannot expect to be able to establish Theorem 6.1.1 by relying solely on these methods. Instead, we will draw from techniques in the theory of automorphic forms.

Remark 6.1.2. One could also consider estimating $N(\lambda; X)$ by parametrising solutions to $F = 0$. To illustrate this, let $F = x_1x_2 - x_3^2 - x_4^2$. Solutions to $F = 0$ in $\mathbf{P}^3$ can be parametrised as $[y_2^2 + y_3^2 : y_1^2 : y_1y_2 : y_1y_3]$, with $[y_1 : y_2 : y_3] \in \mathbf{P}^2$. Thus studying $N(\lambda; X)$ reduces to studying sums of the form

$$\sum_{g \leq X} \sum_{\substack{(y_1, y_2^2+y_3^2)=g \\ y_2^2+y_3^2 \leq gX \\ y_1^2 \leq gX \\ (y_1,y_2,y_3)=1}} \lambda(y_2^2 + y_3^2).$$

The innermost sum can potentially be analysed by the methods developed in [85], although the additional GCD condition makes it a challenging prospect.

We end our introduction by highlighting the key ideas in the proof of Theorem 1.2.5. As is typical when applying the δ-method, an application of Poisson summation in the unweighted variables leads us to study sums that are essentially of the form

$$X \sum_{\substack{\boldsymbol{c}' \in \mathbf{Z}^3 \\ |\boldsymbol{c}'| \ll 1}} \sum_{q \ll X} q^{-\frac{3}{2}} \sum_{n \ll X} \lambda(n) T(A_1 n^2, F^{-1}(0, \boldsymbol{c}'); q) I_q(n, \boldsymbol{c}'). \tag{6.1}$$

Here $I_q(n, \boldsymbol{c}')$ is an exponential integral, F^{-1} is the quadratic form dual to F, and $T(m, n; q)$ is a certain one-dimensional exponential sum of modulus q which, on average, admits square-root cancellation (for fixed m, say). The derivatives $\frac{\partial^j}{\partial n^j} I_q(n, \boldsymbol{c}')$

depend polynomially on X/q, and determining how to control them is one of the main challenges we shall face.

Using Deligne's bound for $\lambda(n)$ and the bound $I_q(n, \boldsymbol{c}') \ll 1$, we see that the sum in (6.1) is $O(X^{2+\varepsilon})$. This will be our starting point, and our objective is to make some saving in the n-sum. In this endeavour, three not unrelated strategies present themselves: exploiting cancellation from sums of Hecke eigenvalues, Mellin inversion, and the Voronoi summation formula. We shall make use of all three methods to successfully analyse the n-sum.

If $F^{-1}(0, \boldsymbol{c}') = 0$ then $T(A_1 n^2, 0; q)$ is essentially a Gauss sum. For fixed q, we shall see that this sum vanishes unless n satisfies certain congruence properties modulo divisors of q. Moreover, $T(A_1 n^2, 0; q)$ is $O(q^{1/2})$ on average, leaving us to get cancellation for sums of the form $\sum_{n \equiv 0 \pmod{d}} \chi(n)\lambda(n) I_q(n, \boldsymbol{c}')$, for χ a Dirichlet character with conductor e, and $[d, e] \mid q$.

On account of the classical bound $\sum_{n \le X} e(\alpha n)\lambda(n) \ll X^{1/2} \log X$ (which is uniform in $\alpha \in \mathbf{R}$), it is natural to try and estimate the n-sum by partial summation. However it appears difficult to derive good bounds for $\partial I_q(n, \boldsymbol{c}')/\partial n$ unless q is large. Instead, we are able to control the Mellin transform of $I_q(n, \boldsymbol{c}')$ by means of a stationary phase argument, and this is one of the main novelties of our approach. The subsequent application of Mellin inversion to estimate the n-sum naturally leads to requiring a subconvexity estimate for twists of $L(s, f)$ by Dirichlet characters, and this allows us to save a small power of X in the n-sum.

On the other hand, if $F^{-1}(0, \boldsymbol{c}')$ does not vanish, Voronoi's formula works well when q is a small power of X. Indeed, if w has support in $[X, 2X]$ and its derivatives satisfy the bound $w^{(j)}(x) \ll_j x^{-j}$, Voronoi's summation formula transforms the sum $\sum \lambda(n) e_q(an) w(n)$ to a 'short' sum of length about q^2/X, when $(a, q) = 1$. However, in our current regime, the derivatives of $I_q(n, \boldsymbol{c}')$ are too large for small q, and we must balance these opposing forces to make a saving in the n-sum. When q is large, partial summation becomes a viable option, and we are able to demonstrate cancellation in the n-sum.

We end by remarking that the methods used in this chapter appear to extend to cover the case when f is not holomorphic. In this case, we have the bound

$$\lambda(n) \ll_{\varepsilon,f} n^{\frac{7}{64}+\varepsilon}$$

due to Kim and Sarnak [60], but this does not affect the analysis significantly. With more effort, one could also establish a similar result for forms with arbitrary level and central character.

Finally, if f is not a cusp form, we will have to account for the appearance of a main term, but the analysis of the error terms will remain unchanged. Although we omit the details, the proof of Theorem 1.2.5 can be suitably modified to give an asymptotic formula for $N(a;X)$ where $a(n) = d(n)$ or $r(n)$.

6.2 Preliminaries

We begin with the following easy consequence of Lemma 3.2.2.

Lemma 6.2.1. *Let $g(x)$ be a smooth function with compact support, and $\lambda(m)$ be the normalised Fourier coefficients of a cusp form of weight k and full level. We then have*

$$\sum_{m\equiv b(\text{mod } q)} \lambda(m)g(m) = \frac{1}{q}\sum_{d|q}\sum_{m=1}^{\infty} \lambda(m)S(b,m;d)\check{g}_d(m), \tag{6.2}$$

where

$$\check{g}_d(m) = \frac{2\pi i^k}{d}\int_0^{\infty} g(x)J_{k-1}\left(\frac{4\pi}{d}\sqrt{xm}\right)\,dx, \tag{6.3}$$

is a Hankel-type transform of g.

Proof. We have

$$\sum_{m\equiv b(\text{mod } q)} \lambda(m)g(m) = \frac{1}{q}\sum_{d|q}\sideset{}{^*}\sum_{r(\text{mod } d)} e_d(-br)\sum_{m=1}^{\infty}\lambda(m)g(m)e_d(rm).$$

The lemma follows by applying the Voronoi summation formula to the inner sum. $\square$

Lemma 6.2.2. *Let $g \in C_0^\infty(\mathbf{R})$ be a smooth function with support in $[1/2, 2]$. Then for any $l \geq 0$ we have*

$$\int_0^\infty g(x) J_{k-1}\left(t\sqrt{x}\right)\, dx \ll_l \min\left\{\|g\|_\infty, \|g\|_{l,1} t^{-(l+1/2)}\right\}.$$

Proof. Denote the left hand side above by $I(t)$. Although this is a standard argument, we present a proof from [33, Proposition 2.3]. Set $\alpha = t^{-2}$. Making the change of variables $x \to \alpha y^2$ we see that

$$I(t) = 2\alpha \int_0^\infty g(\alpha y^2) y J_{k-1}(y)\, dy.$$

Using the recurrence relation (3.2), $(x^k J_k(x))' = x^k J_{k-1}(x)$, and by repeated integration by parts we have

$$I(t) = 2\alpha \int_0^\infty \left\{ \sum_{0 \leq v \leq l} \xi_{v,l} (\alpha y^2)^v g^{(v)}(\alpha y^2) \right\} \frac{J_{k-1+l}(y)}{y^{l-1}}\, dy,$$

for some constants $\xi_{v,l}$. By (3.1) and the fact that $y \asymp \alpha^{-\frac{1}{2}}$, we see that

$$I(t) \ll_l \|g\|_{1,l} t^{-(l+1/2)}.$$

This completes the proof. $\square$

6.2.1 Some facts about L-functions

In this section, we collect some standard facts about L-functions; [56, Chapter 5] is a useful reference. Let f be a Hecke eigenform of weight k and full level with normalised Fourier coefficients $\lambda(n)$ as before. Let χ be a primitive Dirichlet character with conductor D. For $\sigma > 1$ let

$$L(s, f \otimes \chi) = \sum_{n=1}^\infty \frac{\chi(n)\lambda(n)}{n^s}.$$

Then $L(s, f \otimes \chi)$ has analytic continuation to the entire complex plane, satisfies a functional equation, and has an Euler product

$$L(s, f \otimes \chi) = \prod_p \left(1 - \frac{\chi(p)\lambda(p)}{p^s} + \frac{\chi^2(p)}{p^{2s}}\right)^{-1} \tag{6.4}$$

for $\sigma > 1$. Applying the Phragmén-Lindelöf principle in the region $\frac{1}{2} \leq \sigma \leq 1$ to $L(s, f \otimes \chi)$, we get that

$$L(s, f \otimes \chi) \ll_{\varepsilon, f} (D(1 + |t|))^{1-\sigma+\varepsilon}, \tag{6.5}$$

for any $\varepsilon > 0$. When $\sigma = \frac{1}{2}$, we can improve on (6.5). We record the following subconvexity bounds for $L(s, f \otimes \chi)$. It follows from [7] that there exists $A > 0$ such that for all $\varepsilon > 0$ we have

$$L(s, f \otimes \chi) \ll_{\varepsilon, f} D^A (1 + |t|)^{\frac{1}{3}+\varepsilon}. \tag{6.6}$$

Although they are not used here, 'hybrid' subconvexity bounds for $L(s, f \otimes \chi)$ are also known, thanks to the work of Blomer and Harcos [5]: there exists $\delta > 0$ such that for all $\varepsilon > 0$ we have

$$L(s, f \otimes \chi) \ll_{\varepsilon, f} (D(1 + |t|))^{\frac{1}{2}-\delta+\varepsilon}.$$

6.3 Setting up the δ-method

Using Theorem 2.1.4 to detect the equation $F(\boldsymbol{x}) = 0$ we see that

$$N(\lambda, X) = c_Q Q^{-2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a \pmod q} \sum_{\boldsymbol{x} \in \mathbf{Z}^4} \lambda(x_1) e_q(aF(\boldsymbol{x}))\, w\left(\frac{\boldsymbol{x}}{X}\right) h\left(\frac{q}{Q}, \frac{F(\boldsymbol{x})}{Q^2}\right). \tag{6.7}$$

We will take $Q = X$ in our application of the δ-method, since $F(\boldsymbol{x})$ is typically of size X^2 when $\boldsymbol{x}$ is of size X.

6.3.1 Applying the Poisson summation formula

Breaking up the sum in (6.7) into residue classes modulo q we get

$$N(\lambda; X) = c_Q Q^{-2} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{a (\bmod q)} \sum_{\boldsymbol{b} (\bmod q)} e_q(aF(\boldsymbol{b})) \\ \times \sum_{\boldsymbol{x} \equiv \boldsymbol{b} (\bmod q)} \lambda(x_1) w\left(\frac{\boldsymbol{x}}{X}\right) h\left(\frac{q}{Q}, \frac{F(\boldsymbol{x})}{Q^2}\right).$$

Applying Lemma 3.2.1 in the x_2, x_3 and x_4 variables we get that

$$N(\lambda; X) = c_Q X \sum_{q=1}^{\infty} q^{-3} \sum_{\boldsymbol{c}' \in \mathbf{Z}^3} \sideset{}{^*}\sum_{a (\bmod q)} \sum_{\boldsymbol{b} (\bmod q)} e_q(aF(\boldsymbol{b}) + \boldsymbol{b}'.\boldsymbol{c}') \\ \times \sum_{c_1 \equiv b_1 (\bmod q)} \lambda(c_1) I_q(\boldsymbol{c}), \tag{6.8}$$

where if $r = q/X$,

$$I_q(\boldsymbol{c}) = \int_{\mathbf{R}^3} w(c_1/X, \boldsymbol{z}) h(r, F(c_1/X, \boldsymbol{z})) e_r(-\boldsymbol{c}'.\boldsymbol{z})\, d\boldsymbol{z}.$$

By properties of the h-function we see that $q \ll X$, or equivalently, $r \ll 1$.

Set $\boldsymbol{u}' = r^{-1}\boldsymbol{c}'$,

$$F_q(b_1, s) = \sum_{n \equiv b_1 (\bmod q)} \frac{\lambda(n)}{n^s},$$

and

$$I_q(\boldsymbol{c}', s) = \int_{\mathbf{R}^+ \times \mathbf{R}^3} w(\boldsymbol{x}) h(r, F(\boldsymbol{x})) e(-\boldsymbol{u}'.\boldsymbol{x}') x_1^{s-1}\, d\boldsymbol{x}. \tag{6.9}$$

For $\frac{1}{2} \leq \sigma \leq 2$, integrating by parts we see that

$$I_q(\boldsymbol{c}', s) \ll_N |s|^{-N} \left| \int \frac{\partial^N}{\partial x_1^N} \left\{ w(\boldsymbol{x}) h(r, F(\boldsymbol{x})) \right\} x_1^{s+N-1} e(-\boldsymbol{u}'.\boldsymbol{x}') d\boldsymbol{x} \right| \\ \ll_N r^{-1-N} |s|^{-N}, \tag{6.10}$$

by (2.8). For $\sigma > 1$, we have $F_q(b_1, s) \ll 1$, as the Dirichlet series converges absolutely in this region. By the Mellin inversion theorem, we therefore have

$$N(\lambda; X) = c_Q X \sum_{q \ll X} q^{-3} \sum_{\boldsymbol{c}' \in \mathbf{Z}^3} \sideset{}{^*}\sum_{a (\mathrm{mod}\ q)} \sum_{\boldsymbol{b} (\mathrm{mod}\ q)} e_q(aF(\boldsymbol{b}) + \boldsymbol{b}'.\boldsymbol{c}') \\ \times \frac{1}{2\pi i} \int_{(\sigma)} X^s F_q(b_1, s) I_q(\boldsymbol{c}', s)\, ds,$$

whenever $\sigma > 1$.

We end this section by recording an alternate expression for $N(\lambda; X)$. Applying Lemma 3.2.2 to the c_1 variable in (6.8) we see that

$$N(\lambda; X) = c_Q X^2 \sum_{q \ll X} q^{-4} \sum_{\substack{\boldsymbol{c} \in \mathbf{Z}^4 \\ c_1 \geq 1}} \lambda(c_1) \sum_{d | q} S_{d,q}(\boldsymbol{c}) I_{d,q}(\boldsymbol{c}), \tag{6.11}$$

where

$$S_{d,q}(\boldsymbol{c}) = \sideset{}{^*}\sum_{a (\mathrm{mod}\ q)} \sum_{\boldsymbol{b} (\mathrm{mod}\ q)} e_q(aF(\boldsymbol{b}) + \boldsymbol{b'}.\boldsymbol{c}') S(b_1, c_1; d) \tag{6.12}$$

and

$$I_{d,q}(\boldsymbol{c}) = \frac{2\pi i^k}{d} \int_{\mathbf{R}^+ \times \mathbf{R}^3} w\left(\boldsymbol{x}\right) h\left(r, F(\boldsymbol{x})\right) J_{k-1}\left(\frac{4\pi}{d}\sqrt{c_1 X x_1}\right) e\left(-\boldsymbol{u'}.\boldsymbol{x'}\right)\, d\boldsymbol{x}. \tag{6.13}$$

6.4 Integral estimates

6.4.1 First steps

Let

$$w_0(x) = \begin{cases} \exp(-(1 - x^2)^{-1}), & |x| < 1 \\ 0 & |x| \geq 1, \end{cases} \tag{6.14}$$

be a smooth function with compact support and let

$$\gamma(x) = w_0\left(\frac{x}{100 \max_{i=1,2,3,4} |A_i|}\right).$$

Then $\gamma(F(\boldsymbol{x})) \gg 1$ whenever $\boldsymbol{x} \in \operatorname{supp}(w)$. Recall that $r = q/X$ and let

$$g(r, y) = h(r, y)\gamma(y). \tag{6.15}$$

Then g has compact support, and by Lemma 2.1.5 we have the following bound for its Fourier transform,

$$p_r(t) = p(t) = \int_{\mathbf{R}} g(r, y)e(-ty)\,dy \ll_j (r|t|)^{-j}. \tag{6.16}$$

Remark 6.4.1. The above bound shows that $p(t)$ has polynomial decay unless $|t| \ll r^{-1-o(1)}$.

We also record a certain dissection argument due to Heath-Brown [43, Lemma 2]. Let w_0 be as in (6.14), and let $c_0 = \int_{\mathbf{R}} w_0(x)\,dx$. For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{R}^3$ define

$$w_\delta(x_1, \boldsymbol{u}, \boldsymbol{v}) = c_0^{-3} w_0^{(3)}(\boldsymbol{u}) w(x_1, \delta\boldsymbol{u} + \boldsymbol{v}),$$

where

$$w_0^{(3)}(\boldsymbol{u}) = \prod_{i=1}^{3} w_0(u_i).$$

Then

$$\int_{\mathbf{R}^3} w_\delta\left(x_1, \frac{\boldsymbol{x}' - \boldsymbol{y}'}{\delta}, \boldsymbol{y}'\right)\,d\boldsymbol{y}' = \delta^3 w(\boldsymbol{x}). \tag{6.17}$$

Finally, we remind the reader that the test function w is supported in $[1/2, 2]^4$, a fact we will repeatedly make use of. It is crucial to our arguments that w is supported away from the origin.

6.4.2 Estimates for $I_q(\boldsymbol{c})$

Recall that

$$I_q(\boldsymbol{c}) = \int_{\mathbf{R}^3} w(c_1/X, \boldsymbol{z})h(r, F(c_1/X, \boldsymbol{z}))e(-\boldsymbol{u}'.\boldsymbol{z})\,d\boldsymbol{z}.$$

We have the following estimates.

Lemma 6.4.2. $I_q(\boldsymbol{c}) \ll 1$.

Proof. This follows from [43, Lemma 15]. □

Lemma 6.4.3. *Let $N \geq 0$ and suppose that $\boldsymbol{c}' \neq \boldsymbol{0}$. Then*

$$I_q(\boldsymbol{c}) \ll_N \frac{X}{q}|\boldsymbol{c}'|^{-N}.$$

Proof. This follows from [43, Lemma 19]. □

As a consequence of Lemma 6.4.3, we find that $I_q(\boldsymbol{c}) \ll_A X^{-A}$ if $|\boldsymbol{c}'| \gg X^{\varepsilon}$. It remains to examine the behaviour of $I_q(\boldsymbol{c})$ when $|\boldsymbol{c}'| \ll X^{\varepsilon}$.

Lemma 6.4.4. *Let $\varepsilon > 0$. Suppose that $1 \leq |\boldsymbol{c}'| \ll X^{\varepsilon}$. Then, for $j = 0, 1$ we have*

$$\frac{\partial^j}{\partial c_1^j} I_q(\boldsymbol{c}) \ll_{\varepsilon} (r^{-1}|\boldsymbol{u}'|)^{\varepsilon} \left(r^{-j} \left(\frac{c_1}{X^2} \right)^j + \frac{j}{X} \right) |\boldsymbol{u}'|^{-\frac{1}{2}}.$$

Proof. Since $r \ll 1$, we have $|\boldsymbol{u}'| \gg 1$ under the hypotheses of the lemma. By (6.16) we have

$$I_q(\boldsymbol{c}) = \int_{\mathbf{R}} p(t) \int_{\mathbf{R}^3} \widetilde{w}(c_1/X, \boldsymbol{z}) e(tF(c_1/X, \boldsymbol{z}) - \boldsymbol{u}'.\boldsymbol{z})\, d\boldsymbol{z}\, dt,$$

where

$$\widetilde{w}(c_1/X, \boldsymbol{z}) = \frac{w(c_1/X, \boldsymbol{z})}{\gamma(F(c_1/X, \boldsymbol{z}))}.$$

For $j \in \{0, 1\}$,

$$\begin{aligned}\frac{\partial^j}{\partial c_1^j} I_q(\boldsymbol{c}) = &\left(\frac{4\pi i A_1 c_1}{X^2} \right)^j \int t^j p(t) \int \widetilde{w}(c_1/X, \boldsymbol{z}) e(tF(c_1/X, \boldsymbol{z}) - \boldsymbol{u}'.\boldsymbol{z})\, d\boldsymbol{z}\, dt \\ &+ \frac{j}{X} \int p(t) \int \frac{\partial}{\partial c_1} \widetilde{w}(c_1/X, \boldsymbol{z}) e(tF(c_1/X, \boldsymbol{z}) - \boldsymbol{u}'.\boldsymbol{z})\, d\boldsymbol{z}\, dt.\end{aligned}$$

Denote the integrals over $\boldsymbol{z}$ by $I_1(t)$ and $I_2(t)$ respectively. Using Lemma 3.4.3 we have the following bounds for $I_k(t)$:

$$I_k(t) \ll \min\left(1, |t|^{-\frac{3}{2}}\right),$$

and if $|\boldsymbol{u}'| \gg |t|$ then

$$I_k(t) \ll_N |\boldsymbol{u}'|^{-N},$$

for $k=1,2$.

By (6.16), we have the bounds,

$$\int_{|t|\ll|\boldsymbol{u}'|} |t|^j |p(t)| \ll |\boldsymbol{u}'|^{1+j},$$

and

$$\int_{|t|\gg|\boldsymbol{u}'|} |p(t)||t|^{j-\frac{3}{2}}\,dt \ll_j r^{-j}|\boldsymbol{u}'|^{-\frac{1}{2}}.$$

As a result,

$$\frac{\partial^j}{\partial c_1^j} I_q(\boldsymbol{c}) \ll_N \left(\frac{c_1}{X^2}\right)^j \left(|\boldsymbol{u}'|^{1+j-N} + r^{-j}|\boldsymbol{u}'|^{-\frac{1}{2}}\right) + \frac{j}{X}\left(|\boldsymbol{u}'|^{1-N} + |\boldsymbol{u}'|^{-\frac{1}{2}}\right). \tag{6.18}$$

We will see that this is satisfactory for the lemma unless $|\boldsymbol{c}'|$ is essentially $O(1)$. If this is the case, we proceed as follows. By [43, Lemma 15] we have

$$\begin{aligned}
\frac{\partial}{\partial c_1} I_q(\boldsymbol{c}) &= \left(\frac{2A_1c_1}{X^2}\right)\int_{\mathbf{R}^3} w(c_1/X,\boldsymbol{z})\frac{\partial h(r,F(c_1/X,\boldsymbol{z}))}{\partial c_1} e(-\boldsymbol{u}'.\boldsymbol{z}')\,d\boldsymbol{z} + \\
&\qquad \frac{1}{X}\int \frac{\partial w(c_1/X,\boldsymbol{z})}{\partial c_1} h(r,F(c_1/X,\boldsymbol{z}) e(-\boldsymbol{u}'.\boldsymbol{z})\,d\boldsymbol{z} \\
&\ll \left(\frac{2A_1c_1}{X^2}\right)\int r^{-1}\left\{1+\min\left(1,\frac{r^2}{F(c_1/X,\boldsymbol{z})^2}\right)\right\}\,d\boldsymbol{z} + \\
&\qquad \frac{1}{X}\int\left\{1+\min\left(1,\frac{r^2}{F(c_1/X,\boldsymbol{z})^2}\right)\right\}\,d\boldsymbol{z} \\
&\ll r^{-1}\left(\frac{c_1}{X^2}\right) + \frac{1}{X},
\end{aligned} \tag{6.19}$$

by (2.8) and by the observation that the measure of the set of $\boldsymbol{z}$ for which $|F(\frac{c_1}{X},\boldsymbol{z})| \ll \nu$ is $O(\nu)$. Consequently,

$$\frac{\partial^j}{\partial c_1^j} I_q(\boldsymbol{c}) \ll r^{-j}\left(\frac{c_1}{X^2}\right)^j + \frac{j}{X},$$

for $j=0,1$.

We are now in place to finish the proof of the lemma. Suppose first that $|\boldsymbol{u}'| \ll$

$r^{-2\varepsilon/3}$, then $|\boldsymbol{u}'|^{\frac{1}{2}-\varepsilon} \ll r^{-\varepsilon}$. In this case,

$$\frac{\partial^j}{\partial c_1^j} I_q(\boldsymbol{c}) \ll r^{-j}\left(\frac{2c_1}{X^2}\right)^j + \frac{j}{X} \ll (r^{-1}|\boldsymbol{u}'|)^{\varepsilon}\left(r^{-j}\left(\frac{2c_1}{X^2}\right)^j + \frac{j}{X}\right)|\boldsymbol{u}'|^{-\frac{1}{2}}.$$

Suppose next that $|\boldsymbol{u}'| \gg r^{-\frac{2\varepsilon}{3}}$ then choosing N large enough in (6.18) we get

$$\frac{\partial^j}{\partial c_1^j} I_q(\boldsymbol{c}) \ll \left(r^{-j}\left(\frac{c_1}{X^2}\right)^j + \frac{j}{X}\right)|\boldsymbol{u}'|^{-\frac{1}{2}}.$$

This completes the proof of the lemma. □

Remark 6.4.5. The reader should compare the preceding result to [43, Lemma 22].

6.4.3 Estimates for $I_q(\boldsymbol{c}', s)$ and $I_{d,q}(\boldsymbol{c})$

In this section we will denote the complex variable $s = \sigma + it$, where σ and t are real. Recall that

$$I_q(\boldsymbol{c}', s) = \int_{\mathbf{R}^+ \times \mathbf{R}^3} w(\boldsymbol{x}) h(r, F(\boldsymbol{x})) e(-\boldsymbol{u}'.\boldsymbol{x}') x_1^{s-1}\, d\boldsymbol{x}$$

and

$$I_{d,q}(\boldsymbol{c}) = \frac{2\pi i^k}{d} \int_{\mathbf{R}^+ \times \mathbf{R}^3} w(\boldsymbol{x})\, h(r, F(\boldsymbol{x}))\, J_{k-1}\left(\frac{4\pi}{d}\sqrt{c_1 X x_1}\right) e(-\boldsymbol{u}'.\boldsymbol{x}')\, d\boldsymbol{x}.$$

6.4.3.1 First estimates

The following 'trivial' bounds follow from [43, Lemma 15] and the bound (3.1). Our task for the rest of the section will be to improve upon them.

Lemma 6.4.6. *Let $\frac{1}{2} \leq \sigma \leq 2$. We have*

$$I_q(\boldsymbol{c}', s) \ll 1$$

and

$$I_{d,q}(\boldsymbol{c}) \ll \frac{1}{d}\left(1 + \frac{\sqrt{c_1 X}}{d}\right)^{-\frac{1}{2}}.$$

Next we will give estimates for $I_q(\boldsymbol{c}', s)$ and $I_{d,q}(\boldsymbol{c})$ in the spirit of Lemmas 6.4.3 and 6.4.4.

By Fourier inversion we may write

$$I_q(\boldsymbol{c}', s) = \int_{\mathbf{R}} p(\alpha) \int_{\mathbf{R}^+ \times \mathbf{R}^3} \widetilde{w}(\boldsymbol{x}) e(\alpha F(\boldsymbol{x}) - \boldsymbol{u}'.\boldsymbol{x}') x_1^{s-1} \, d\boldsymbol{x} \, d\alpha, \tag{6.20}$$

where

$$\widetilde{w}(\boldsymbol{x}) = \frac{w(\boldsymbol{x})}{\gamma(F(\boldsymbol{x}))}. \tag{6.21}$$

Let

$$\Psi(\boldsymbol{x}') = \alpha F(0, \boldsymbol{x}') - \boldsymbol{u}'.\boldsymbol{x}'. \tag{6.22}$$

Therefore,

$$I_q(\boldsymbol{c}', s) = \int_{\mathbf{R}} p(\alpha) \int_{\mathbf{R}^+} x_1^{s-1} e(\alpha A_1 x_1^2) \int_{\mathbf{R}^3} \widetilde{w}(\boldsymbol{x}) e(\Psi(\boldsymbol{x}')) \, d\boldsymbol{x}' \, dx_1 \, d\alpha.$$

If $|\alpha| \ll |\boldsymbol{u}'|$, then $\nabla\Psi(\boldsymbol{x}') \gg |\boldsymbol{u}'|$, and as a result, the integral over $\boldsymbol{x}'$ is $O(|\boldsymbol{u}'|^{-N})$ by Lemma 3.4.1. Since the integral over $\boldsymbol{x}'$ is trivially $O(1)$, we have the bound

$$I_q(\boldsymbol{c}', s) \ll_N |\boldsymbol{u}'|^{-N} \int_{|\alpha| \ll |\boldsymbol{u}'|} |p(\alpha)| \, d\alpha + \int_{|\alpha| \gg |\boldsymbol{u}'|} |p(\alpha)| \, d\alpha.$$

Therefore, by (6.16) we have established the following result.

Lemma 6.4.7. *Suppose that* $\mathbf{c}' \neq \mathbf{0}$ *and* $\frac{1}{2} \leq \sigma \leq 2$. *Then*

$$I_q(\boldsymbol{c}', s) \ll \min\left\{1, r^{-1} |\boldsymbol{c}'|^{-N}\right\}.$$

As a result, $I_q(\boldsymbol{c}', s) \ll_A X^{-A}$ *unless* $|\boldsymbol{c}'| \ll X^{\varepsilon}$.

Turning to $I_{d,q}(\boldsymbol{c})$ we begin by using (6.15) to get

$$I_{d,q}(\boldsymbol{c}) = \frac{2\pi i^k}{d} \int_{\mathbf{R}^+ \times \mathbf{R}^3} \widetilde{w}(\boldsymbol{x}) g(r, F(\boldsymbol{x})) J_{k-1}\left(4\pi \frac{\sqrt{c_1 X x_1}}{d}\right) d\boldsymbol{x}.$$

Next we apply Lemma 6.2.2 with

$$\psi(x_1) = \psi_{\boldsymbol{x}'}(x_1) = \tilde{w}(\boldsymbol{x})g(r, F(\boldsymbol{x}))e(-\boldsymbol{u}'.\boldsymbol{x}'),$$

treated as a function in the variable x_1, and $t = 4\pi(c_1X)^{\frac{1}{2}}/d$ to obtain the bound

$$I_{d,q}(\boldsymbol{c}) \ll_N \frac{1}{d}\left(4\pi\frac{\sqrt{c_1X}}{d}\right)^{1/2} \|\psi\|_{N,1}\left(4\pi\frac{\sqrt{c_1X}}{d}\right)^{-N}.$$

Since $\frac{\partial^n}{\partial y^n}h(r,y) \ll r^{-1-n}$, we have the bound $\|\psi\|_{N,1} \ll_N r^{-1-N}$, which gives us

$$I_{d,q}(\boldsymbol{c}) \ll_N (dr)^{-1}\left(4\pi\frac{\sqrt{c_1X}}{d}\right)^{1/2}\left(\frac{X}{q}\frac{d}{\sqrt{c_1X}}\right)^{N}.$$

Hence we have the following

Lemma 6.4.8. *For all $N \geq 0$ we have*

$$I_{d,q}(\boldsymbol{c}) \ll_N (dr)^{-1}\left(\frac{d^{\frac{1}{2}}}{(c_1X)^{\frac{1}{4}}}\right)\min\left\{\left(\frac{X}{q}\frac{d}{\sqrt{c_1X}}\right)^{N-1}, |\boldsymbol{c}'|^{-N}\right\}$$

As a result, $I_{d,q}(\boldsymbol{c}) \ll_A X^{-A}$ whenever $c_1 \gg X^{1+\varepsilon}/(q/d)^2$, or $|\boldsymbol{c}'| \gg X^{\varepsilon}$.

Proof. The first bound follows from the preceding discussion. The second follows from [43, Lemma 19], and the bound $J_{k-1}(x) \ll (1+x)^{-1/2}$ and taking $N > 2A/\varepsilon$. $\square$

Finally, we also record the bound

$$I_{d,q}(\boldsymbol{c}) \ll \frac{1}{d}(1+\sqrt{c_1X}/d)^{-\frac{1}{2}}, \tag{6.23}$$

which follows from (2.8).

6.4.3.2 A stationary phase argument

Our goal in this section will be to establish the following results.

Lemma 6.4.9. *Let $\varepsilon > 0$ and suppose that $\mathbf{c}' \neq 0$. For $\sigma > 0$ we have*

$$I_q(\boldsymbol{c}', s) \ll_\varepsilon \frac{1}{|s|} X^\varepsilon.$$

Lemma 6.4.10. *Let $\varepsilon > 0$ and suppose that $\boldsymbol{c}' \neq \mathbf{0}$. Then*

$$I_{d,q}(\boldsymbol{c}) \ll_\varepsilon \frac{1}{d}\left(1 + \frac{\sqrt{c_1 X}}{d}\right)^{-\frac{1}{2}} \left(\frac{q}{X}\right) X^\varepsilon.$$

Integrating $I_q(\boldsymbol{c}', s)$ by parts we get

$$\begin{aligned} I_q(\boldsymbol{c}', s) &= -\frac{1}{s}\int \frac{\partial \widetilde{w}(\boldsymbol{x})}{\partial x_1} g(r, F(\boldsymbol{x})) e(-\boldsymbol{u}'.\boldsymbol{x}') x_1^s \, d\boldsymbol{x} \\ &\quad - \frac{2A_1}{s}\int \widetilde{w}(\boldsymbol{x}) \frac{\partial}{\partial x_1} g(r, F(\boldsymbol{x})) e(-\boldsymbol{u}'.\boldsymbol{x}') x_1^{s+1} \, d\boldsymbol{x} \\ &= -\frac{1}{s}(I_1 + I_2), \end{aligned} \tag{6.24}$$

say. Lemma 6.4.6 applied to the test function $\partial \widetilde{w}(\boldsymbol{x})/\partial x_1$ shows that

$$I_1 \ll 1$$

and arguing as in (6.19) we see that

$$I_2 \ll r^{-1}. \tag{6.25}$$

Hence it suffices to focus on I_2: our task is to remove the factor r^{-1} in the bound for I_2.

Next, we remark that if $c_1 \ll d^2/X$, Lemma 6.4.10 follows from [43, Lemma 22], simply by repeating Heath-Brown's argument with the weight function

$$w(\boldsymbol{x}) J_{k-1}\left(4\pi \frac{\sqrt{c_1 X x_1}}{d}\right).$$

Hence it suffices to focus on the range $c_1 \gg d^2/X$. In this range, we use the integral representation of the Bessel function given by (3.3).

Remark 6.4.11. If we were to run Heath-Brown's argument in the complementary range $c_1 \gg d^2/X$, then *a priori* we would only have the weaker bound

$$I_{d,q}(\boldsymbol{c}) \ll_\varepsilon \frac{1}{d}\frac{q}{X}X^\varepsilon.$$

In order to save the extra factor of $\frac{\sqrt{c_1 X}}{d}$ it is imperative that we use the decay properties of the Bessel function.

By (6.20) we have

$$I_2 = \frac{4\pi i A_1}{s}\int_{\mathbf{R}} \alpha p(\alpha)\int_{\mathbf{R}^+\times\mathbf{R}^3} \widetilde{w}(\boldsymbol{x})x_1^{\sigma+1}e(\alpha A_1 x_1^2 + \tfrac{t}{2\pi}\log x_1 + \Psi(\boldsymbol{x}'))\,d\boldsymbol{x}\,d\alpha, \quad (6.26)$$

where $\Psi(\boldsymbol{x}')$ is as in (6.22), and

$$\begin{aligned} I_{d,q}(\boldsymbol{c}) = &\frac{2\pi i^k}{d}\int p(\alpha)\int \widetilde{w}(\boldsymbol{x})W_{k-1}\left(4\pi\frac{\sqrt{c_1 X x_1}}{d}\right)e(\alpha A_1 x_1^2 + 2\tfrac{\sqrt{c_1 X x_1}}{d} + \Psi(\boldsymbol{x}'))\,d\boldsymbol{x}\,d\alpha \\ &+ \frac{2\pi i^k}{d}\int p(\alpha)\int \widetilde{w}(\boldsymbol{x})\overline{W_{k-1}\left(4\pi\frac{\sqrt{c_1 X x_1}}{d}\right)}e(\alpha A_1 x_1^2 - 2\tfrac{\sqrt{c_1 X x_1}}{d} + \Psi(\boldsymbol{x}'))\,d\boldsymbol{x}\,d\alpha. \end{aligned} \quad (6.27)$$

We will treat I_2 and $I_{d,q}(\boldsymbol{c})$ in a unified way, and our approach is modeled on the proof of [43, Lemma 22]. We will use Heath-Brown's argument in the $\boldsymbol{x}'$ variable, and the second derivative estimate for exponential integrals to handle the integral over x_1. To execute this argument, we begin by defining a certain class of functions $\mathcal{H}$ that will be of interest to us.

Definition 6.4.12. Let $u(x) : \mathbf{R} \to \mathbf{R}$ be a smooth function and let $v(\boldsymbol{x})$ be a smooth function with compact support in $[1/2, 2]^4$. We say that the pair $(u, v) \in \mathcal{H}$ if the following properties are satisfied.

1. If $u \neq 0$, then there exists a real number U with the property that

$$|U| \ll_j |u^{(j)}(x)| \ll_j |U|$$

for $x \in [1/2, 2]$ and each $j \geq 0$, and

2. For all $j_i \geq 0$ we have
$$\frac{\partial^{j_1+\ldots+j_4} v(\boldsymbol{x})}{\partial x_1^{j_1} \ldots \partial x_4^{j_4}} \ll 1.$$

We will first prove the following version of [43, Lemma 20].

Lemma 6.4.13. *Let u and v be functions such that $(u, v) \in \mathcal{H}$. Let $\Psi(\boldsymbol{x}')$ be as in (6.22). For $j = 0, 1$ let*

$$\mathcal{I}_j = \int \alpha^j p(\alpha) \int v(\boldsymbol{x}) e(\alpha A_1 x_1^2 + u(x_1) + \Psi(\boldsymbol{x}'))\, d\boldsymbol{x}\, d\alpha. \tag{6.28}$$

Suppose that there exists $R \geq 1$ such that $R^3 \leq |\boldsymbol{u}'| \leq r^{-1} R$. Then there exists a smooth real valued function v_1 with support in $[1/2, 2]$, and a real number ω satisfying $|\boldsymbol{u}'| \ll_F |\omega| \ll_F |\boldsymbol{u}'|$ such that the following holds

$$\mathcal{I}_j \ll_{v,F} r^{-1-j} R^{-N} + R^3 r^{\frac{1}{2}-j} \left| \int v_1(x) e(A_1 \omega x^2 + u(x))\, dx \right|.$$

Proof. Applying (6.17) to $v(\boldsymbol{x})$ we get that

$$\begin{aligned}\mathcal{I}_j = \delta^{-3} &\int \alpha^j p(\alpha) \times \\ &\int\int v_\delta \left(x_1, \frac{\boldsymbol{x}' - \boldsymbol{y}}{\delta}, \boldsymbol{y} \right) e(\alpha A_1 x_1^2 + u(x_1) + \Psi(\boldsymbol{x}'))\, d\boldsymbol{x}\, d\boldsymbol{y}\, d\alpha.\end{aligned}$$

Let $\boldsymbol{x}' = \boldsymbol{y} + \delta \boldsymbol{z}$. By virtue of v being compactly supported, we see that $|\boldsymbol{y}| \ll 1$, and we arrive at the inequality

$$\begin{aligned}\mathcal{I}_j \leq &\int\int |\alpha^j p(\alpha)| \times \\ &\left| \int v_{\boldsymbol{y}}(x_1, \boldsymbol{z}) e(\alpha A_1 x_1^2 + u(x_1) + \Psi(\boldsymbol{y} + \delta \boldsymbol{z}))\, dx_1\, d\boldsymbol{z} \right| d\alpha\, d\boldsymbol{y}.\end{aligned}$$

with $v_{\boldsymbol{y}}(x_1, \boldsymbol{z}) = v_\delta(x_1, \boldsymbol{z}, \boldsymbol{y})$. Henceforth, we will take $\delta = |\boldsymbol{u}'|^{-\frac{1}{2}}$.

Let $\boldsymbol{y} = (y_2, y_3, y_4)$. As in the proof of [43, Lemma 22], we say that a pair $(\boldsymbol{y}, \alpha)$ is 'good', if

$$|\nabla \Psi(\boldsymbol{0})| = |\boldsymbol{u}'|^{-\frac{1}{2}} |2\alpha(A_2 y_2, A_3 y_3, A_4 y_4) - \boldsymbol{u}'| \geq R \max\{|\alpha|/|\boldsymbol{u}'|, 1\},$$

and that $(\boldsymbol{y},\alpha)$ is 'bad' otherwise. If $(\boldsymbol{y},\alpha)$ is 'good' then Lemma 3.4.1 shows that

$$\int\left|\int v_{\boldsymbol{y}}(x_1,\boldsymbol{z})e(\Psi(\boldsymbol{y}+\delta\boldsymbol{z}))\,d\boldsymbol{z}\right|\,dx_1 \ll_N R^{-N}.$$

For the 'bad' pairs we will bound the x_1 integral using a stationary phase argument and bound the $\boldsymbol{z}$ integral trivially.

Suppose that $(\boldsymbol{y},\alpha)$ is bad. Since $|\boldsymbol{u}'|^{-\frac{1}{2}} \ll R^{-\frac{3}{2}}$, observe that $|\boldsymbol{y}| \gg_F 1$. Moreover, $|\boldsymbol{y}| \ll 1$, trivially, and as a result, we see that $|\boldsymbol{u}'| \ll |\alpha| \ll |\boldsymbol{u}'|$. Therefore, if $(\boldsymbol{y},\alpha)$ is bad, we find that

$$|2\alpha(A_2y_2,A_3y_3,A_4y_4)-\boldsymbol{u}'| \ll R|\boldsymbol{u}'|^{\frac{1}{2}}. \tag{6.29}$$

As a result, the measure of the set of bad pairs is $O(R^3|\boldsymbol{u}'|^{-\frac{3}{2}})$. Taking the supremum over the bad pairs $(\boldsymbol{y},\alpha)$ and over $\boldsymbol{z}$ we get that there exists $|\omega| \asymp |\boldsymbol{u}'|$, and vectors $\boldsymbol{y}_0$, $\boldsymbol{z}_0$ such that

$$I_2 \ll_N r^{-2}R^{-N} + R^3 r^{\frac{1}{2}-j}\left|\int v_{\boldsymbol{y}_0}(x_1,\boldsymbol{z}_0)e(A_1\omega x^2+u(x))\,dx\right|,$$

since $\int|p(\alpha)|\,d\alpha \ll r^{-1}$. This completes the proof of the lemma. □

To prove Lemmas 6.4.9 and 6.4.10 we will apply the preceding lemma to the pairs of functions $(\frac{t}{2\pi}\log x_1, \widetilde{w}(\boldsymbol{x}))$ and $(2\sqrt{c_1Xx_1}/d, W_{k-1}(4\pi\sqrt{c_1Xx_1}/d)\widetilde{w}(\boldsymbol{x}))$ respectively. It is easy to verify that they lie in $\mathcal{H}$ (in the latter case, use (3.4)). To evaluate the ensuing integrals over x_1, we need the following results.

Lemma 6.4.14. *Let w have compact support in $[1/2,2]$, and suppose that $A \neq 0$. Then for all $N \geq 0$ we have*

$$\int w(x)e(Ax^2+B\log x)\,dx \ll_N \max\left\{|A|^{-\frac{1}{2}},|A|^{-N}\right\}. \tag{6.30}$$

Proof. Let $\Psi(x)=Ax^2+B\log x$. Suppose that $|B|\geq 8|A|$. We then have $|\Psi'(x)|\geq B$ in the support of w. And $|\Psi''(x)|=|2A-\frac{B}{2x^2}| \ll |B|$, $|\Psi^{(n)}(x)|=\left|\frac{B}{n!x^n}\right| \ll_n |B|$ for all $n \geq 3$. By [43, Lemma 10] we get

$$\int w(x)e(\Psi(x))\,dx \ll |B|^{-N} \ll |A|^{-N}.$$

Suppose next that $|B| \leq \frac{1}{8}|A|$. In this case, $|\Psi'(x)| \geq |A|$ in the support of w, and we also see that $|\Psi^{(n)}(x)| \ll_n |A|$ for all $n \geq 2$. Consequently,

$$\int w(x)e(\Psi(x))\,dx \ll |A|^{-N}$$

in this case as well.

Suppose finally that $\frac{1}{8}|A| \leq |B| \leq 8|A|$. If AB is positive, then it is easy to see that $|\Psi'(x)| \gg |A|$, and as a result $\int w(x)e(\Psi(x))\,dx \ll |A|^{-N}$. Finally, suppose that AB is negative. Then in this case $|\Psi''(x)| = |2A - \frac{B}{2x^2}| \gg |A|$, and we appeal to Lemma 3.4.2. This concludes the proof. $\square$

Lemma 6.4.15. *Let* $\alpha \geq 1$. *Define*

$$I(\alpha, A) = \int_{1/2}^{2} u(x)W_{k-1}(\alpha\sqrt{x})e(Ax^2 + \tfrac{\alpha}{2\pi}\sqrt{x})\,dx,$$

where $u(x)$ *is a smooth function with support inside* $[1/2, 2]$ *and* $A\alpha \neq 0$. *Then we have*

$$I(\alpha, A) \ll (\alpha|A|)^{-\frac{1}{2}}.$$

Proof. The proof closely follows that of Lemma 6.4.14. We are led to study the integral

$$-\int_{1/2}^{2}(u(x)W_{k-1}(\alpha\sqrt{x}))'\int_{1/2}^{x} e(Ay^2 + \tfrac{\alpha}{2\pi}\sqrt{y})\,dy\,dx.$$

Arguing exactly as in the proof of Lemma 6.4.14, and using (3.4) we get that

$$I(\alpha, A) \ll (A\alpha)^{-\frac{1}{2}}.$$

$\square$

Proof of Lemmas 6.4.9 and 6.4.10. Recall that we may assume that $1 \leq |\boldsymbol{c}'| \ll X^{\varepsilon}$

and $d^2/X \ll c_1 \ll X^{1+\varepsilon}/(q/d)^2$. All that remains to do is to choose R. Suppose first that $|\boldsymbol{u}'| \ll r^{-\varepsilon/2}$. Then, $|\boldsymbol{u}'|^{1-\varepsilon} \ll r^{-\varepsilon}$. In this case, we make use of the trivial bounds (6.25) and (6.23) to get,

$$I_2 \ll r^{-1} \ll (r^{-1}|\boldsymbol{u}'|)^{\varepsilon} r^{-1}|\boldsymbol{u}'|^{-1} \ll_{\varepsilon} (r^{-1}|\boldsymbol{u}'|)^{\varepsilon}|\boldsymbol{c}'|^{-1},$$

and

$$I_{d,q}(\boldsymbol{c}) \ll \frac{1}{d}\left(\frac{\sqrt{c_1 X}}{d}\right)^{-\frac{1}{2}} \ll (r^{-1}|\boldsymbol{u}'|)^{\varepsilon}|\boldsymbol{u}'|^{-1}\frac{1}{d}\left(\frac{\sqrt{c_1 X}}{d}\right)^{-\frac{1}{2}} \ll_{\varepsilon} \frac{X^{\varepsilon}}{d}\left(\frac{\sqrt{c_1 X}}{d}\right)^{-\frac{1}{2}}\left(\frac{q}{X}\right).$$

If, on the other hand, $|\boldsymbol{u}'| \gg r^{-\varepsilon/2}$, choose $R = (r^{-1}|\boldsymbol{u}|')^{\varepsilon/12}$. By taking N sufficiently large, we get from Lemmas 6.4.13, 6.4.14 and 6.4.15 that

$$I_2 \ll_N R^3 + r^{-2}R^{-N} \ll_{\varepsilon} (r^{-1}|\boldsymbol{u}'|)^{\varepsilon}|\boldsymbol{c}'|^{-1} \ll_{\varepsilon} X^{\varepsilon}$$

and

$$I_{d,q}(\boldsymbol{c}) \ll (dr)^{-1}R^{-N} + \frac{r}{d}\left(\frac{\sqrt{c_1 X}}{d}\right)^{-\frac{1}{2}} R^3 \ll_{\varepsilon} \frac{X^{\varepsilon}}{d}\left(\frac{\sqrt{c_1 X}}{d}\right)^{-\frac{1}{2}}\left(\frac{q}{X}\right).$$

This completes the proof of the lemma. $\square$

6.5 Exponential sums

We begin by establishing certain multiplicativity results for the exponential sums that we will encounter in the proof of Theorem 1.2.5. Let $S_{d,q}(\boldsymbol{c})$ be the exponential sum in (6.12). We have

Lemma 6.5.1. *Suppose that $d = u_1 u_2$ and $q = v_1 v_2$ with $(u_1 v_1, u_2 v_2) = 1$. Then the following holds,*

$$\begin{aligned} S_{d,q}(\boldsymbol{c}) &= S_{u_1,v_1}(\overline{u_2}^2 c_1, \overline{v_2}\boldsymbol{c}')S_{u_2,v_2}(\overline{u_1}^2 c_1, \overline{v_1}\boldsymbol{c}') \\ &= S_{u_1,v_1}(v_2\overline{u_2^2} c_1, \boldsymbol{c}')S_{u_2,v_2}(v_1\overline{u_1^2} c_1, \boldsymbol{c}'). \end{aligned}$$

Proof. Let $a = v_2 a_1 + v_1 a_2$ where a_i run modulo v_i. Let $\boldsymbol{b} = v_2\overline{v_2}\boldsymbol{s} + v_1\overline{v_1}\boldsymbol{t}$ where s (respectively t) runs modulo v_1 (respectively v_2). Then

$$e_q(aF(\boldsymbol{b}) + \boldsymbol{b}'.\boldsymbol{c}') = e_{v_1}(a_1F(\boldsymbol{s}) + \boldsymbol{s}'.\overline{v_2}\boldsymbol{c}')e_{v_2}(a_2F(\boldsymbol{t}) + \boldsymbol{t}'.\overline{v_1}\boldsymbol{c}').$$

Also,

$$\begin{aligned} S(b_1, c_1; u_1u_2) &= S(\overline{u_2}s_1, \overline{u_2}c_1; u_1)S(\overline{u_1}t_1, \overline{u_1}c_1; u_2) \\ &= S(s_1, \overline{u_2}^2c_1; u_1)S(t_1, \overline{u_1}^2c_1; u_2). \end{aligned}$$

This gives us the first multiplicativity statement. For the second, replace $\boldsymbol{s}$ (resp. $\boldsymbol{t}$) by $v_2\boldsymbol{s}$ (resp. $v_1\boldsymbol{t}$). $\square$

Let

$$A_q(\boldsymbol{c}) = \sum_{a (\mathrm{mod}\ q)}^{*} \sum_{\boldsymbol{b}' (\mathrm{mod}\ q)} e_q(aF(c_1, \boldsymbol{b}') + \boldsymbol{b}'.\boldsymbol{c}'). \tag{6.31}$$

Lemma 6.5.2. *Let $q = v_1v_2$ with $(v_1, v_2) = 1$. We have*

$$\begin{aligned} A_q(\boldsymbol{c}) &= A_{v_1}(c_1, \overline{v_2}\boldsymbol{c}')A_{v_2}(c_1, \overline{v_1}\boldsymbol{c}') \\ &= A_{v_1}(\overline{v_2}^2c_1, \boldsymbol{c}')A_{v_2}(\overline{v_1}^2c_1, \boldsymbol{c}'). \end{aligned}$$

Proof. The proof is similar to the proof above: let $d = 1$ and write a and $\boldsymbol{b}'$ as in Lemma 6.5.1. We have,

$$e_q(aF(c_1, \boldsymbol{b}') + \boldsymbol{b}'.\boldsymbol{c}') = e_{v_1}(a_1F(n, \boldsymbol{x}') + \boldsymbol{x}'.\overline{v_2}\boldsymbol{c}')e_{v_2}(a_2F(n, \boldsymbol{y}') + \boldsymbol{y}'.\overline{v_1}\boldsymbol{c}').$$

Therefore,

$$A_q(\boldsymbol{c}) = A_{v_1}(c_1, \overline{v_2}\boldsymbol{c}')A_{v_2}(c_1, \overline{v_1}\boldsymbol{c}').$$

The lemma follows by replacing $\boldsymbol{x}'$ by $v_2\boldsymbol{x}'$, $\boldsymbol{y}'$ by $v_1\boldsymbol{y}'$, and by replacing a_1 by $\overline{v_2}^2a_1$ and a_2 by $\overline{v_1}^2a_2$. $\square$

6.5.1 Evaluation of $S_q(n)$

Set

$$S_q(n) = A_q(n, \mathbf{0}). \tag{6.32}$$

Lemma 6.5.2 shows that $S_{q_1q_2}(n) = S_{q_1}(n)S_{q_2}(n)$ whenever $(q_1, q_2) = 1$. Therefore, it suffices to evaluate $S_q(n)$ at prime powers $q = p^k$. In doing so, we will encounter the following exponential sums, and it will be useful to have their evaluation at hand. For $p > 2$ define

$$\begin{aligned} \mathcal{S}^-(p^k, n) &= \sideset{}{^*}\sum_{a (\mathrm{mod}\ p^k)} e_{p^k}(an^2) \left(\frac{a}{p}\right), \text{ and} \\ \mathcal{S}^+(p^k, n) &= \sideset{}{^*}\sum_{a (\mathrm{mod}\ p^k)} e_{p^k}(an^2) = c_{p^k}(n^2). \end{aligned} \tag{6.33}$$

Lemma 6.5.3. *Suppose that p is an odd prime. Then we have*

$$\mathcal{S}^-(p^k, n) = \begin{cases} 0 & k \textit{ is even,} \\ \epsilon_p p^{k-\frac{1}{2}} \mathbf{1}_{v_p(n^2)=k-1} & \textit{otherwise.} \end{cases}$$

$$\mathcal{S}^+(p^k, n) = \begin{cases} \varphi(p^k)\mathbf{1}_{p^{k/2}|n} & \textit{if } k \textit{ is even,} \\ p^k \left(\mathbf{1}_{p^{\frac{k+1}{2}}|n} - \frac{\mathbf{1}_{p^{\frac{k-1}{2}}|n}}{p}\right) & \textit{if } k \textit{ is odd.} \end{cases}$$

Proof. For $\mathcal{S}^-$, we write $a = u + pv$ and

$$\begin{aligned} \mathcal{S}^-(p^k, n) &= \sideset{}{^*}\sum_{u (\mathrm{mod}\ p)} \left(\frac{u}{p}\right) e_{p^k}(un^2) \sum_{v (\mathrm{mod}\ p^{k-1})} e_{p^{k-1}}(vn^2) \\ &= p^{k-1}\mathbf{1}_{p^{k-1}|n^2} \sideset{}{^*}\sum_{u (\mathrm{mod}\ p)} \left(\frac{u}{p}\right) e_p(un^2/p^{k-1}) \\ &= \epsilon_p p^{k-\frac{1}{2}} \mathbf{1}_{p^{k-1}\|n^2} \left(\frac{n^2/p^{k-1}}{p}\right). \end{aligned}$$

By definition, $\mathcal{S}^+(p^k, n)$ is Ramanujan's sum, and its evaluation is well-known. $\square$

For the remainder of the section we will assume that coefficients $A_1, \ldots, A_4$ satisfy the following condition.

Definition 6.5.4. [Condition A_0] Let l_1, l_2, l_3 and l_4 be non-zero integers, and let $q = \prod_{p^{k_p} \| q}$. We say that the tuple $(q; l_1, l_2, l_3, l_4)$ satisfies *Condition* A_0 if for each odd prime $p \mid q$, we have $k_p \geq \max\{v_p(l_1), v_p(l_2), v_p(l_3), v_p(l_4)\}$, if $p \nmid \Delta$, and if $k_p \geq \max\{2 + v_p(l_1), v_p(l_2), v_p(l_3), v_p(l_4)\}$ for $p \mid \Delta$. If $p = 2$, we require that $k_2 \geq 3 + \max_{1 \leq i \leq 4} v_2(l_i)$.

For a prime p and $1 \leq i \leq 4$, let $a_i(p) = v_p(A_i)$. Define the following product of Jacobi symbols,

$$J(p^k) = J_\Delta(p^k) = \prod_{i=2}^{4} \left(\frac{\overline{A_1}^{-1} \overline{A_i}}{p^{k-a_i}} \right), \tag{6.34}$$

if $p \neq 2$, and

$$J(2^k) = \prod_{2 \leq i \leq 4} \left(\frac{2^{k-a_i}}{\overline{A_1}^{-1} \overline{A_i}} \right), \tag{6.35}$$

where $\overline{A_i} = A_i/(p^{a_i}, A_i)$. $J(q)$ is then defined multiplicatively for arbitrary q.

6.5.1.1 Evaluation of $S_q(n)$ for odd q

To state our result on the evaluation of $S_q(n)$ for odd q we define the following invariant. Let

$$\epsilon(p^k) = \epsilon_\Delta(p^k) = \begin{cases} 1 & \text{if each } k - a_i \pmod 2 \text{ has the same parity for } 2 \leq i \leq 4 \\ \epsilon_p^2 & \text{otherwise,} \end{cases} \tag{6.36}$$

and extend the definition of $\epsilon(q)$ to odd q by multiplicativity.

Lemma 6.5.5. *Let $q = p^k$, and $p \neq 2$ and suppose that $(q, A_1, \ldots, A_4)$ satisfies Condition A_0. If $k - a_2 - a_3 - a_4$ is even,*

$$S_q(n) = q^{\frac{3}{2}} J(p^k) p^{a_1} c_{p^{k-a_1}}(n^2) \epsilon(p^k) \prod_{i=2}^{4} p^{\frac{a_i}{2}}.$$

If $k-a_2-a_3-a_4$ is odd, then $S_q(n)$ vanishes unless $k-a_1$ is odd, in which case,

$$S_q(n) = \mathbf{1}_{v_p(n^2)=k-a_1-1} p^{-\frac{1}{2}} q^{\frac{5}{2}} J(p^k)\epsilon(p^k) \prod_{i=2}^{4} p^{\frac{a_i}{2}}.$$

Proof. We have

$$S_q(n) = \sideset{}{^*}\sum_{a (\mathrm{mod}\ p^k)} e_q(aA_1n^2) \prod_{i=2}^{4} \sum_{b_i (\mathrm{mod}\ q)} e_q(aA_ib_i^2).$$

Applying Lemma 3.3.1 to each of the sums over b_i we get that

$$\begin{aligned} S_q(n) &= p^{\frac{3k}{2}} \prod_{i=2}^{4} \left(\frac{\overline{A_i}}{p^{k-a_i}}\right) p^{\frac{a_i}{2}} \epsilon_{p^{k-a_i}} \sideset{}{^*}\sum_{a (\mathrm{mod}\ p^k)} e_{p^k}(aA_1n^2) \prod_{i=2}^{4} \left(\frac{a}{p^{k-a_i}}\right) \\ &= p^{\frac{3k}{2}} p^{a_1} \mathcal{S}^{\pm}(p^{k-a_1}, n) J(p^k) \prod_{i=2}^{4} \epsilon_{p^{k-a_i}} p^{\frac{a_i}{2}}, \end{aligned}$$

depending on the parity of $k-a_2-a_3-a_4 \pmod 2$. If $k-a_2-a_3-a_4$ is even, then by Lemma 6.5.3

$$S_q(n) = q^{\frac{3}{2}} J(p^k) p^{a_1} c_{p^{k-a_1}}(n^2) \prod_{i=2}^{4} p^{\frac{a_i}{2}} \epsilon_{p^{k-a_i}}.$$

If $k-a_2-a_3-a_4$ is odd, we have

$$\begin{aligned} S_q(n) &= q^{\frac{3}{2}} p^{a_1} J(p^k) \epsilon_p \prod_{i=2}^{4} \epsilon_{p^{k-a_i}} p^{\frac{a_i}{2}} \mathcal{S}^{-}(p^{k-a_1}, n) \\ &= \mathbf{1}_{v_p(n^2)=k-a_1-1} J(p^k) p^{-\frac{1}{2}} q^{\frac{5}{2}} \epsilon_p \prod_{i=2}^{4} p^{\frac{a_i}{2}} \epsilon_{p^{k-a_i}}. \end{aligned}$$

This completes the proof of the lemma. $\square$

6.5.1.2 Evaluation of $S_{p^k}(n)$ for $p=2$

Let $q = 2^k$. By Lemma 3.3.1,

$$\begin{aligned} S_q(n) &= \sum_{a(\text{mod } 2^k)}^{*} e_{2^k}(aA_1n^2)\prod_{i=2}^{4} 2^{a_i} \sum_{b_i(\text{mod } 2^{k-a_i})} e_{2^{k-a_i}}\left(a\overline{A_i}b_i^2\right) \\ &= (1+i)^3 2^{\frac{3k+a_2+a_3+a_4}{2}} \sum_{a(\text{mod } 2^k)}^{*} e_{2^k}(aA_1n^2)\prod_{2\le i\le 4} \epsilon_{\overline{aA_i}}^{-1}\left(\frac{2^{k-a_i}}{a\overline{A_i}}\right) \\ &= J(2^k)2^{a_1+\frac{3k+a_2+a_3+a_4}{2}} \sum_{v(\text{mod } 2^{k-2-a_1})} e_{2^{k-2}}(vn^2)\times \\ &\quad \sum_{u(\text{mod } 4)}^{*} e_{2^{k-2-a_1}}(un^2)\prod_{2\le i\le 4}(1+i)\epsilon_{\overline{uA_1}^{-1}\overline{A_i}}^{-1}\left(\frac{2^{k-a_i}}{u+4v}\right). \end{aligned} \tag{6.37}$$

Suppose first that $k-a_2-a_3-a_4$ is even, in which case $\prod_{2\le i\le 4}\left(\frac{2^{k-a_i}}{\cdot}\right)=1$. In this case, the sum over v vanishes unless $n^2 \equiv 0 \pmod{2^{k-2-a_1}}$. Let T denote the sum over u. We have

$$T = (1+i)^3 \sum_{u(\text{mod } 4)}^{*} e_4(un^2/2^{k-2-a_1})\prod_{2\le i\le 4}\epsilon_{\overline{uA_1}^{-1}\overline{A_i}}^{-1}.$$

Define the invariant

$$\gamma(\Delta) = \prod_{2\le i\le 4}\epsilon_{\overline{A_1}^{-1}\overline{A_i}}^{-1},$$

which is a fourth root of unity that depends on the coefficients of F. A simple calculation reveals that

$$\prod_{2\le i\le 4}\epsilon_{\overline{A_1}^{-1}\overline{A_i}}^{-1}\prod_{2\le i\le 4}\epsilon_{\overline{3A_1}^{-1}\overline{A_i}}^{-1} = i.$$

Let

$$\delta^+(n,2^k) = \frac{(1+i)^3}{4}\begin{cases}\left(\gamma(\Delta)+\frac{i}{\gamma(\Delta)}\right) & \text{if } \frac{n^2}{2^{k-2-a_1}} \equiv 0 \pmod 4 \\ i\left(\gamma(\Delta)-\frac{i}{\gamma(\Delta)}\right) & \text{if } \frac{n^2}{2^{k-2-a_1}} \equiv 1 \pmod 4 \\ -\left(\gamma(\Delta)+\frac{i}{\gamma(\Delta)}\right) & \text{if } \frac{n^2}{2^{k-2-a_1}} \equiv 2 \pmod 4. \end{cases}$$

Then $T = 4\delta^+(n, 2^k)$. Substituting back into (6.37) we get

$$S_q(n) = \mathbf{1}_{2^{k-2-a_1}|n^2}\delta^+(n,k)J(2^k)2^{\frac{5k+a_2+a_3+a_4}{2}}.$$

Suppose finally that $k - a_2 - a_3 - a_4$ is odd. Proceeding as in (6.37), but writing $a = u + 8v$, we get that

$$S_q(n) = 2^{\frac{3k+a_2+a_3+a_4}{2}} J(2^k) \sum_{v(\text{mod } 2^{k-3})} e_{2^{k-3-a_1}}(vn^2)\times$$
$$\sum^*_{u(\text{mod } 8)} e_8(un^2/2^{k-3-a_1}) \left(\frac{2}{u}\right) \prod_{2\le i\le 4} (1+i)\epsilon^{-1}_{\overline{uA_1}^{-1}\overline{A_i}}$$

The sum over v vanishes unless $2^{k-3-a_1} \mid n^2$ - in which case, up to a factor of $(1+i)^3$, the inner sum is

$$e_8(n^2/2^{k-3-a_1})\gamma(\Delta) - \frac{ie_8(3n^2/2^{k-3-a_1})}{\gamma(\Delta)} - e_8(5n^2/2^{k-3-a_1})\gamma(\Delta) + \frac{ie_8(7n^2/2^{k-3-a_1})}{\gamma(\Delta)},$$

which vanishes unless $2^{k-3-a_1}\|n^2$. Therefore, this forces $n^2/2^{k-3-a_1} \equiv 1 \pmod 8$. Let

$$\delta^-(n, 2^k) = \frac{-1}{\sqrt{2}}\left(\gamma(\Delta) + \frac{1}{\gamma(\Delta)}\right). \tag{6.38}$$

Then we see that the u-sum evaluates to $8\delta^-(n, 2^k)$. As a result,

$$S_q(n) = \mathbf{1}_{v_2(n^2)=k-3-a_1} J(2^k)\delta^-(n, 2^k)2^{\frac{5k+a_2+a_3+a_4}{2}}.$$

Lemma 6.5.6. *Let $q = 2^k$, and notation as above. Suppose that $(q; A_1, \ldots, A_4)$ satisfies Condition A_0. Then*

$$S_q(n) = 2^{\frac{5k+a_2+a_3+a_4}{2}} J(2^k) \begin{cases} \delta^+(n, 2^k)\mathbf{1}_{2^{k-2-a_1}|n^2} & \text{if } k - a_2 - a_3 - a_4 \text{ is even} \\ \delta^-(n, 2^k)\mathbf{1}_{v_2(n^2)=k-3-a_1} & \text{if } k - a_2 - a_3 - a_4 \text{ is odd.} \end{cases}$$

6.5.1.3 A description of $S_q(n)$ for general q

Having evaluated $S_q(n)$ at prime-powers, we will now record a qualitative description of $S_q(n)$ for general q. Write $q = q_{odd}q_{even}$ where $(q_{odd}, q_{even}) = 1$ and $q_{even} \mid 2^\infty$. Write $q_{odd} = q_1q_2$, where q_1 is composed entirely of primes $p \mid q$ such that $k_p - \sum_{i=2}^4 a_i(p)$ is odd, and q_2 is composed entirely of primes such that $k_p - \sum_{i=2}^4 a_i(p)$ is even. By Lemma 6.5.5 we see that $S_{q_1}(n)$ vanishes $k_p - a_1(p)$ is odd, for each $p \mid q_1$. To this end define

$$\iota(q_1) = \begin{cases} 1 & k_p - a_1(p) \text{ is odd for each } p \mid q_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $s(q)$ denote the squarefree kernel of an integer q. By Lemma 6.5.2 we have

$$S_q(n) = S_{q_{even}}(n)S_{q_1}(n)S_{q_2}(n).$$

For an integer q define

$$\varrho(q) = \prod_{\substack{p^k \| q \\ k \text{ is odd}}} p^{\frac{k+1}{2}} \prod_{\substack{p^k \| q \\ k \text{ is even}}} p^{\frac{k}{2}}$$

and let

$$\widetilde{q_1} = \varrho\left(\frac{q_1}{s(q_1)(q_1, A_1)}\right).$$

Invoking Lemma 6.5.5, we see that $S_{q_1}(n)$ vanishes unless $n = \widetilde{q_1}m$ with $(m, s(q_1)) = 1$. Let $\chi^0_{s(q_1)}$ be the principal character modulo $s(q_1)$. We have,

$$S_{q_1}(n) = \iota(q_1)\epsilon(q_1)J(q_1)\prod_{i=2}^4 (q_1, A_i)^{\frac{1}{2}} \mathbf{1}_{\widetilde{q_1}|n}\chi^0_{s(q_1)}(n/\widetilde{q_1})\frac{q_1^{\frac{5}{2}}}{s(q_1)^{\frac{1}{2}}}. \tag{6.39}$$

Similarly,

$$S_{q_2}(n) = q_2^{\frac{3}{2}}\epsilon(q_2)J(q_2)\prod_{i=2}^4 (q_2, A_i)^{\frac{1}{2}}(q_2, A_1)c_{q_2/(q_2,A_1)}(n^2). \tag{6.40}$$

To give an explicit description of $c_{q_2/(q_2,A_1)}(n^2)$, decompose $q_2 = q_3q_4$, with

$$q_3 = \prod_{\substack{p^k \| q_2 \\ k-a_1(p) \text{ is odd}}} p^k$$
$$q_4 = \prod_{\substack{p^k \| q_2 \\ k-a_1(p) \text{ is even}}} p^k.$$

Then it follows from the definitions of q_3 and q_4, and Lemma 6.5.3 that

$$\begin{aligned} c_{q_4/(q_4,A_1)}(n^2) &= \varphi\left(\frac{q_4}{(q_4,A_1)}\right)\mathbf{1}_{\sqrt{q_4/(q_4,A_1)}|n} \\ &= \frac{q_4}{(q_4,A_1)}\prod_{p|q_4/(q_4,A_1)}\left(1-\frac{1}{p}\right)\mathbf{1}_{\varrho(q_4/(q_4,A_1))|n} \end{aligned}$$

and

$$\begin{aligned} c_{q_3/(q_3,A_1)}(n^2) &= \prod_{p^k\|q_3/(q_3,A_1)} p^k\left(\mathbf{1}_{p^k|n^2}-\frac{\mathbf{1}_{p^{k-1}|n^2}}{p}\right) \\ &= \frac{q_3}{(q_3,A_1)}\sum_{d|\frac{q_3}{(q_3,A_1)}}\frac{\mu(d)}{d}\mathbf{1}_{\frac{q_3/(q_3,A_1)}{d}|n^2} \\ &= \frac{q_3}{(q_3,A_1)}\sum_{d|\frac{q_3}{(q_3,A_1)}}\frac{\mu(d)}{d}\mathbf{1}_{\varrho\left(\frac{q_3/(q_3,A_1)}{d}\right)|n}. \end{aligned}$$

Substituting back into (6.40), we obtain

$$S_{q_2}(n) = \epsilon(q_2)J(q_2)\prod_{i=2}^{4}(q_2,A_i)^{\frac{1}{2}}\frac{\varphi(q_4/(q_4,A_1))}{q_4/(q_4,A_1)}q_2^{\frac{5}{2}}\sum_{d|\frac{q_3}{(q_3,A_1)}}\frac{\mu(d)}{d}\mathbf{1}_{\varrho\left(\frac{q_2/(q_2,A_1)}{d}\right)|n}.$$

Combining with Lemma 6.5.6 to evaluate $S_{q_{even}}(n)$, we obtain the following result.

Proposition 6.5.7. *Suppose that $(q; A_1, \ldots, A_4)$ satisfies condition A_0. Then there exist integers $s_q(F)$, $\theta \mid q$ and $\kappa \mid q$ such that $S_q(n) = \mathbf{1}_{\theta|n}\mathbf{1}_{(n/\theta,\kappa)=1}s_q(F)$. Moreover, $s_q(F)$ is independent of n and it satisfies the bound $|s_q(F)| \ll_\Delta q^{\frac{5}{2}}$. In addition, $s_q(F)$ is multiplicative in q, and if q is square-free then $s_q(F) \ll_\Delta q^2$. We also have $\theta \gg_\Delta \varrho\left(\frac{q}{s(q)(q,A_1)}\right)$ and $\kappa \ll_\Delta s(q)$.*

Proof. The existence of θ and κ, and the lower bound for θ follow from (6.39), (6.40) and Lemma 6.5.6. It is clear that $S_q(n) \ll q^{\frac{5}{2}}$, and this gives our bound for $|s_q(F)|$. The multiplicativity of $s_q(F)$ follows from the multiplicativity of $S_q(n)$.

Finally, suppose that q is square-free. Since $(p, A_1, \ldots, A_4)$ satisfies Condition A_0 for each $p \mid q$, we see that $q_2 = 1$. As a result, $|s_q(F)| \ll q^2$, by (6.39). This completes the proof of the proposition. □

6.5.2 Exponential sums in the case where $F^{-1}(0, \boldsymbol{c}') = 0$ and $\boldsymbol{c}' \neq \mathbf{0}$

Having evaluated $S_q(n) = A_q(n, \mathbf{0})$ explicitly, we will now relate it to the more general sum $A_q(n, \boldsymbol{c}')$ with $F^{-1}(0, \boldsymbol{c}') = 0$.

Lemma 6.5.8. *Let $\boldsymbol{c}' \neq \mathbf{0} \in \mathbf{Z}^3$ and let $n \in \mathbf{N}$. Let p be a prime and $q = p^k$. For $2 \leq i \leq 4$ suppose that $c_i \equiv 0 \pmod{p^{v_p(A_i)}}$, and that $F^{-1}(0, \boldsymbol{c}') = 0$. Let $A_q(n, \boldsymbol{c}')$ be as in* (6.31)*. Suppose that (q, A_2, A_3, A_4) satisfies Condition A_0 (Definition 6.5.4). Then*

$$A_q(n, \boldsymbol{c}') = A_q(n, \mathbf{0}) = S_q(n).$$

Proof. Let $a_i = v_p(A_i)$, as before. Then the sum over $\boldsymbol{b}'$ in (6.31) is

$$\prod_{i=2}^{4} p^{a_i} \sum_{b_i \pmod{p^{k-a_i}}} e_{p^{k-a_i}}(aA_i/p^{a_i} b_i^2 + b_i c_i/p^{a_i}).$$

To ease notation, let $A_i' = A_i/p^{a_i}$ and let $c_i' = c_i/p^{a_i}$. If $p \neq 2$, by Lemma 3.3.1 the above expression evaluates to

$$\prod_{i=2}^{4} p^{\frac{k+a_i}{2}} \epsilon_{p^{k-a_i}} \left(\frac{aA_i'}{p^{k-a_i}} \right) e_{p^{k-a_i}}(-\overline{4aA_i'} c_i'^2), \tag{6.41}$$

since by hypothesis $(A_i', p) = 1$. Make a change of variables $\overline{a} \to -4A_2'A_3'A_4'b$, and observe that for $i = 2, 3, 4$ we have $e_{p^{k-a_i}}(-\overline{4aA_i'} c_i'^2) = e_{p^{k-a_i}}(b \prod_{\substack{2 \leq j \leq 4 \\ i \neq j}} A_j' c_i'^2)$. Consequently, the expression in (6.41) is

$$\prod_{i=2}^{4} p^{\frac{k+a_i}{2}} \epsilon_{p^{k-a_i}} \left(\frac{-4b \prod_{\substack{2\le j\le 4 \\ i\neq j}} A'_j}{p^{k-a_i}} \right) e_{p^k}(p^{a_2} A'_3 A'_4 c_2'^2 + p^{a_3} A'_2 A'_4 c_3'^2 + p^{a_4} A'_2 A'_3 c_4'^2).$$

However, since $F^{-1}(0, \boldsymbol{c}') = A_3 A_4 c_2^2 + \ldots + A_2 A_3 c_4^2 = 0$, we have $p^{a_2} A'_3 A'_4 c_2'^2 + p^{a_3} A'_2 A'_4 c_3'^2 + p^{a_4} A'_2 A'_3 c_4'^2 = 0$. Consequently, the exponential factor above is $= 1$, and we see that $A_q(n, \boldsymbol{c}')$ is independent of $\boldsymbol{c}'$ and this completes the proof for odd p. A similar argument works when $p = 2$.

□

6.5.3 Auxillary estimates

Recall the sum $S_{d,q}(\boldsymbol{c})$ from (6.12). We begin by recording a version of [43, Lemma 28].

Lemma 6.5.9. *Let $q = p^t$, $d = p^\delta$ with $t \geq 2$ and $\delta \leq t$. Suppose that $p \nmid 2\Delta$ and $F^{-1}(0, \boldsymbol{c}') \neq 0$. Then $S_{d,q}(\boldsymbol{c})$ vanishes unless $p \mid F^{-1}(0, \boldsymbol{c}')$.*

Proof. In the expression

$$S_{d,q}(\boldsymbol{c}) = \sideset{}{^*}\sum_{z (\text{mod } p^\delta)} e_{p^\delta}(c_1 \overline{z}) \sideset{}{^*}\sum_{a (\text{mod } p^t)} \sum_{\boldsymbol{b} (\text{mod } p^t)} e_{p^t}(aF(\boldsymbol{b}) + \boldsymbol{b}'.\boldsymbol{c}' + p^{t-\delta} b_1 z)$$

set $a = u + pv$ to see that

$$\begin{aligned} S_{d,q}(\boldsymbol{c}) = & \sideset{}{^*}\sum_{z (\text{mod } p^\delta)} e_{p^\delta}(c_1 \overline{z}) \sideset{}{^*}\sum_{u (\text{mod } p)} \sum_{\boldsymbol{b} (\text{mod } q)} e_{p^t}(uF(\boldsymbol{b}) + \boldsymbol{b}.(p^{t-\delta} z, \boldsymbol{c}')) \times \\ & \sum_{v (\text{mod } p^{t-1})} e_{p^{t-1}}(vF(\boldsymbol{b})) \\ = & \, p^{t-1} \sideset{}{^*}\sum_{z (\text{mod } p^\delta)} e_{p^\delta}(c_1 \overline{z}) \sideset{}{^*}\sum_{u (\text{mod } p)} \sum_{\substack{\boldsymbol{b} (\text{mod } p^t) \\ F(\boldsymbol{b}) \equiv 0 (\text{mod } p^{t-1})}} e_{p^t}(uF(\boldsymbol{b}) + \boldsymbol{b}.(p^{t-\delta} z, \boldsymbol{c}')). \end{aligned}$$

Writing $\boldsymbol{b} = \boldsymbol{x} + p^{t-1}\boldsymbol{y}$, we get that

$$\begin{aligned}S_{d,q}(\boldsymbol{c}) = p^{t-1} &\sum_{z (\mathrm{mod}\ p^\delta)}^{*} e_{p^\delta}(c_1 \overline{z}) \times \\ &\sum_{u (\mathrm{mod}\ p)}^{*} \sum_{\substack{\boldsymbol{x} (\mathrm{mod}\ p^{t-1}) \\ F(\boldsymbol{x}) \equiv 0 (\mathrm{mod}\ p^{t-1})}} e_{p^t}(uF(\boldsymbol{x}) + p^{t-\delta} x_1 z + \boldsymbol{x}'.\boldsymbol{c}') \times \\ &\sum_{\boldsymbol{y} (\mathrm{mod}\ p)} e_p(\boldsymbol{y}.(u \nabla F(\boldsymbol{x}) + (p^{2t-\delta-1} z, \boldsymbol{c}'))).\end{aligned}$$

As $2t \geq 2 + \delta$, the sum over $\boldsymbol{y}$ vanishes unless $\nabla F(\boldsymbol{x}) \equiv -\overline{u}(0, \boldsymbol{c}') \pmod{p}$. Since $p \nmid 2\Delta$, this is the same as the condition $\boldsymbol{x} \equiv -\overline{2u} M^{-1}(0, \boldsymbol{c}') \pmod{p}$, where M is the matrix corresponding to the quadratic form F. Observe that this forces $F(\boldsymbol{x}) \equiv \overline{4u^2} F^{-1}(0, \boldsymbol{c}') \pmod{p}$. Consequently, the sum over $\boldsymbol{x}$ vanishes unless $F^{-1}(0, \boldsymbol{c}') \equiv 0 \pmod{p}$, and the lemma follows. $\square$

Let

$$T_q(r) = \sum_{a (\mathrm{mod}\ q)}^{*} \sum_{\boldsymbol{b} (\mathrm{mod}\ q)} e_q(aF(\boldsymbol{b}) + b_1 r + \boldsymbol{b}'.\boldsymbol{c}').$$

and

$$T_q = \sum_{r (\mathrm{mod}\ q)} |T_q(r)|. \tag{6.42}$$

In the proof of Theorem 1.2.5, we will need good control on the average order of T_q. We have

Lemma 6.5.10. *Suppose that $F^{-1}(0, \boldsymbol{c}') \neq 0$ and $|\boldsymbol{c}'| \ll X^\varepsilon$. Then*

$$\sum_{q \leq X} T_q \ll_\varepsilon X^{4+\varepsilon}.$$

Proof. Observe that T_q is multiplicative in q. Write $q = uv$ where u is square-free and v is square-full. Let $N = 2|\Delta||F^{-1}(0, \boldsymbol{c}')|$. Further factorise $v = v_1 v_2$, with the property that $(v_1, N) = 1$ and $p \mid N$ for any prime p that divides v_2. Thus we are led to estimating T_u, T_{v_1} and T_{v_2} individually.

If $p \nmid 2\Delta$, it follows from [43, Lemma 26] that

$$T_p = p^2 \sum_{r_p (\text{mod } p)} |c_p(F^{-1}(r_p, \boldsymbol{c}'))| \leq 3p^3.$$

Furthermore, if $p \mid 2\Delta$, observe that $T_p \ll_F 1$. Hence we have

$$T_u \ll u^3 3^{\omega(u)}.$$

By [43, Lemma 25] we see that

$$T_{v_2} \ll v_2^4.$$

To deal with T_{v_1} we make the following claim.

Suppose that $p \nmid 2\Delta$ and that $F^{-1}(0, \boldsymbol{c}') \neq 0$. Let $r \pmod{p^t}$ and $p \mid r$. Then we claim that $T_{p^t}(r) = 0$ unless $p \mid F^{-1}(0, \boldsymbol{c}')$.

To see this, we argue as in the proof of Lemma 6.5.9 to see that

$$T_{p^t}(r) = p^{t+3} \sideset{}{^*}\sum_{u (\text{mod } p)} \sum_{\boldsymbol{x} (\text{mod } p^{t-1})} e_{p^t}(uF(\boldsymbol{x}) + rx_1 + \boldsymbol{x}'.\boldsymbol{c}'),$$

where the $\boldsymbol{x}$-sum is also subject to the conditions $F(\boldsymbol{x}) \equiv 0 \pmod{p^{t-1}}$ and $2M\boldsymbol{x} \equiv -\overline{u}(r, \boldsymbol{c}') \pmod{p}$, and M is the matrix associated to the quadratic form F. It is then easy to see that if $p \mid r$ then $p \mid x_1$, and this in turn implies that $p \mid F^{-1}(0, \boldsymbol{c}')$, as claimed.

It now follows that

$$T_{v_1} = \sideset{}{^*}\sum_{r (\text{mod } v_1)} |T_{v_1}(r)|.$$

By applying Lemma 3.3.1 to each term in T_{v_1} we get

$$\begin{aligned} T_{v_1} &= v_1^2 \sideset{}{^*}\sum_{r (\text{mod } v_1)} |c_{v_1}(\overline{A_1} r^2 + F^{-1}(0, \boldsymbol{c}'))| \\ &\leq v_1^2 \sideset{}{^*}\sum_{r (\text{mod } v_1)} (v_1, r^2 + F^{-1}(0, \boldsymbol{c}')) \ll_\varepsilon X^\varepsilon v_1^3, \end{aligned}$$

since $|\boldsymbol{c}'| \ll X^{\varepsilon}$. As a result,

$$\sum_{q \leq X} T_q \ll X^{\varepsilon} \sum_{\substack{v_2 \leq X \\ p|v_2 \implies p|N}} v_2^4 \sum_{uv_1 \leq X/v_2} (uv_1)^3 \ll_{\varepsilon} X^{4+\varepsilon}$$

since

$$\sum_{\substack{v \leq X \\ p|v \implies p|N}} 1 \ll_{\varepsilon} (NX)^{\varepsilon}.$$

This completes the proof of the lemma. □

Remark 6.5.11. Notice that we do not need the condition $F^{-1}(0, \boldsymbol{c}') \neq 0$ to estimate the sum over the square-free part. However, we have used this fact to restrict the number of terms in the v-sum. Without this observation, Lemma 6.5.10 would only hold with the weaker upper bound $O(X^{\frac{9}{2}+\varepsilon})$.

Next, we analyse the sum $S_{d,q}(\boldsymbol{c})$. Observe that Lemma 6.5.1 shows that it suffices to consider the case where $q = p^k$ is a prime power. Lemma 6.5.9 shows that for $(p, 2\Delta) = 1$, if $p^2 \mid q$ then $S_{d,q}(\boldsymbol{c})$ vanishes unless $p \mid |F^{-1}(0, \boldsymbol{c}')|$. If $d = 1$, then $S_{1,q}(\boldsymbol{c}) = S_{1,q}(0, \boldsymbol{c}')$.

Let $d = p^{\delta}$ and $q = p^{\kappa}$, with $\delta \leq \kappa$. Recall that

$$S_{d,q}(\boldsymbol{c}) = \sideset{}{^*}\sum_{a (\bmod p^{\kappa})} \sum_{\boldsymbol{b} (\bmod p^{\kappa})} e_{p^{\kappa}}(aF(\boldsymbol{b}) + \boldsymbol{b}'.\boldsymbol{c}') S(b_1, c_1; p^{\delta}).$$

Suppose first that $p \neq 2$. For $1 \leq i \leq 4$, let $p^{a_i} = (A_i, p^{\kappa})$. By Lemma 3.3.1 we see that $S_{d,q}(\boldsymbol{c})$ vanishes unless $c_i \equiv 0 \pmod{p^{a_i}}$ and in this case,

$$\begin{aligned} S_{d,q}(\boldsymbol{c}) = p^{\frac{3k+a_2+a_3+a_4}{2}} &\prod_{i=2}^{4} \epsilon_{p^{\kappa-a_i}} \left(\frac{A_i/p^{a_i}}{p^{\kappa-a_i}} \right) \times \\ &\sideset{}{^*}\sum_{a (\bmod p^{\kappa})} \prod_{i=2}^{4} \left(\frac{a}{p^{\kappa-a_i}} \right) e_{p^{\kappa-a_i}}(-\overline{4aA_i/p^{a_i}}(c_i/p^{a_i})^2) \\ &\sum_{b_1 (\bmod p^{\kappa})} e_{p^{\kappa}}(aA_1 b_1^2) S(b_1, c_1; p^{\delta}). \end{aligned} \tag{6.43}$$

If $(p, \Delta) = 1$, and $d = q = p$, we have $a_i = 0$, and notice that the sum over b_1

in (6.43) is

$$= \sum_{x(\text{mod } p)}^{*} e_p(c_1\overline{x}) \sum_{b_1(\text{mod } p)} e_p(aA_1b_1^2 + b_1x)$$
$$= \epsilon_p p^{\frac{1}{2}} \left(\frac{aA_1}{p}\right) \sum_{x(\text{mod } p)}^{*} e_p(c_1\overline{x} - \overline{4aA_1}x^2).$$

Consequently,

$$S_{p,p}(\boldsymbol{c}) = p^2 \left(\frac{\Delta}{p}\right) \sum_{a,x(\text{mod } p)}^{*} e_p(c_1\overline{x} - \overline{4a}F^{-1}(x, \boldsymbol{c}'))$$
$$= p^2 \left(\frac{\Delta}{p}\right) \left\{ \varphi(p) \sum_{\substack{x(\text{mod } p) \\ F^{-1}(x,\boldsymbol{c}')\equiv 0(\text{mod } p)}}^{*} e_p(c_1\overline{x}) - \sum_{\substack{x(\text{mod } p) \\ F^{-1}(x,\boldsymbol{c}')\not\equiv 0(\text{mod } p)}}^{*} e_p(c_1\overline{x}) \right\}.$$

Hence $|S_{p,p}(\boldsymbol{c})| \le 3p^3$.

We can also handle these sums in greater generality. Suppose that $p \ne 2$. By Lemma 3.3.1, the sum over b_1 in (6.43) vanishes unless $\delta \le \kappa - a_1$, and in this case, the sum is

$$\begin{aligned} &= \sum_{x(\text{mod } p^\delta)}^{*} e_{p^\delta}(c_1\overline{x}) \sum_{b_1(\text{mod } p^\kappa)} e_{p^\kappa}(aA_1b_1^2 + p^{\kappa-\delta}b_1x) \\ &= \epsilon_{p^{k-a_1}} p^{\frac{k+a_1}{2}} \left(\frac{aA_1/p^{a_1}}{p^{\kappa-a_1}}\right) \sum_{x(\text{mod } p^\delta)}^{*} e_{p^\delta}(c_1\overline{x} - \overline{4aA_1/p^{a_1}}p^{\kappa-\delta-a_1}x^2). \end{aligned} \tag{6.44}$$

If $\delta \le \frac{\kappa-a_1}{2}$, we see that

$$S_{d,q}(\boldsymbol{c}) = p^{2k+\frac{a_1+a_2+a_3+a_4}{2}} \prod_{i=1}^{4} \epsilon_{p^{k-a_i}} \left(\frac{A_i/p^{a_i}}{p^{k-a_i}}\right) c_{p^\delta}(c_1) \times$$
$$\sum_{a(\text{mod } p^\kappa)}^{*} \left(\frac{A_1/p^{a_1}}{p^{\kappa-a_1}}\right) \prod_{i=2}^{4} \left(\frac{A_i/p^{a_i}}{p^{\kappa-a_i}}\right) e_{p^{\kappa-a_i}}(-\overline{4aA_i/p^{a_i}}(c_i/p^{a_i})^2)$$
$$\ll_\Delta p^{3k+\delta} \ll q^{7/2}.$$

If $\delta > \frac{\kappa-a_1}{2}$, clearing denominators in (6.44), we get

$$\begin{aligned} S_{d,q}(\boldsymbol{c}) = {} & p^{3\kappa-\delta-a_1+\frac{a_1+a_2+a_3+a_4}{2}} \prod_{i=1}^{4} \epsilon_{p^{k-a_i}} \left(\frac{A_i/p^{a_i}}{p^{k-a_i}}\right) \times \\ & \sideset{}{^*}\sum_{a (\bmod q)} \left(\frac{A_1/p^{a_1}}{p^{\kappa-a_1}}\right) \prod_{i=2}^{4} \left(\frac{A_i/p^{a_i}}{p^{\kappa-a_i}}\right) e_{p^{\kappa-a_i}}(-\overline{4aA_i/p^{a_i}}(c_i/p^{a_i})^2) \times \\ & \sideset{}{^*}\sum_{x (\bmod p^{2\delta-\kappa+a_1})} e_{p^{2\delta-\kappa+a_1}}(c_1\overline{x} - \overline{4aA_1/p^{k-a_1}}x^2) \\ \ll_\Delta {} & q^{\frac{7}{2}}, \end{aligned}$$

by [13, Lemma 3.1] applied to the sum over x. A similar analysis holds when $p = 2$, except we have the slightly worse bound (see [13, Lemma 3.2])

$$S_{d,q}(\boldsymbol{c}) \ll 2^{\frac{15\kappa}{4}}$$

in this case. Therefore, we have shown the following

Lemma 6.5.12. *Suppose that $d \mid q = p^\kappa$. If $\kappa \geq 2$, $F^{-1}(0, \boldsymbol{c}') \neq 0$, and $p \nmid 2\Delta F^{-1}(0, \boldsymbol{c}')$, then $S_{d,q}(\boldsymbol{c}) = 0$. If $\kappa \geq 2$, $p = 2$ we have*

$$S_{d,q}(\boldsymbol{c}) \ll_\Delta q^{\frac{7}{2}+\frac{1}{4}},$$

and if $p \neq 2$ and $\kappa \geq 2$, we have

$$S_{d,q}(\boldsymbol{c}) \ll_\Delta q^{\frac{7}{2}}.$$

Finally, if $(p, 2\Delta) = 1$ and $q = p$. Then $S_{1,p}(\boldsymbol{c}) \ll p^2(p, F^{-1}(0, \boldsymbol{c}'))$, and $|S_{p,p}(\boldsymbol{c})| \leq 3p^3$. If $p \mid 2\Delta$, then, $S_{1,p}(\boldsymbol{c}) \ll_\Delta 1$ and $S_{p,p}(\boldsymbol{c}) \ll_\Delta 1$.

6.6 Proof of Theorem 1.2.5

It follows from (6.8) and Lemma 6.4.3 that for any $\varepsilon > 0$,

$$N(\lambda; X) = c_Q X \sum_{q \ll X} q^{-3} \sum_{|\boldsymbol{c}'| \ll X^{\varepsilon}} \sideset{}{^*}\sum_{a (\mathrm{mod}\ q)} \sum_{\boldsymbol{b} (\mathrm{mod}\ q)} e_q(aF(\boldsymbol{b}) + \boldsymbol{b}'.\boldsymbol{c}')$$
$$\times \sum_{c_1 \equiv b_1 (\mathrm{mod}\ q)} \lambda(c_1) I_q(\boldsymbol{c}) + O(1).$$

Our task now is to show that the right hand side is $o(X^2)$. The analysis of the exponential sum is predicated on the vanishing or non-vanishing of $F^{-1}(0, \boldsymbol{c}')$. Define the sets

$$\mathcal{C}_0 = \left\{ \boldsymbol{c}' \in \mathbf{Z}^3, |\boldsymbol{c}'| \ll X^{\varepsilon} : F^{-1}(0, \boldsymbol{c}') = 0 \right\},$$
$$\mathcal{C}_1 = \left\{ \boldsymbol{c}' \in \mathbf{Z}^3, |\boldsymbol{c}'| \ll X^{\varepsilon} : F^{-1}(0, \boldsymbol{c}') \neq 0 \right\}.$$

For $i = 0, 1$ let $N^{(i)}(\lambda; X)$ denote the contribution from $\boldsymbol{c}' \in \mathcal{C}_i$. We will show that there exists a $\delta > 0$ such that $N^{(i)}(\lambda; X) \ll X^{2-\delta}$. We start with $N^{(0)}(\lambda; X)$.

6.6.1 Contribution from $N^{(0)}(\lambda; X)$

Let $|\boldsymbol{c}'| \ll X^{\varepsilon}$ such that $F^{-1}(0, \boldsymbol{c}') = 0$. Recall the sum $A_q(\boldsymbol{c})$ from (6.31). Set

$$N^{(0)}(\lambda, \boldsymbol{c}'; X) = \sum_{q \ll X} q^{-3} \sum_{c_1 = 1}^{\infty} \lambda(c_1) A_q(\boldsymbol{c}) I_q(\boldsymbol{c}).$$

Then $N^{(0)}(\lambda; X) = c_Q X \sum_{\boldsymbol{c}' \in \mathcal{C}_0} N^{(0)}(\lambda, \boldsymbol{c}'; X)$.

We begin by writing $q = rs$, a product of coprime integers, as follows. Recalling Condition A_0 (Definition 6.5.4), let

$$r = \prod_{\substack{p^k \| q \\ (p^k; A_1, \ldots, A_4) \text{ satisfies} \\ \text{Condition } A_0}} p^k$$

be the greatest divisor of q that satisfies Condition A_0. By Lemma 6.5.2 we have

$A_q(\boldsymbol{c}) = A_r(\overline{s}^2c_1, \boldsymbol{c}')A_s(\overline{r}^2c_1, \boldsymbol{c}')$. Lemma 3.3.1 shows that $A_r(\boldsymbol{c})$ vanishes unless $c_i \equiv 0 \pmod{p^{v_p(A_i)}}$, for $2 \le i \le 4$, so without loss of generality, we may assume that $\boldsymbol{c}'$ satisfies this condition. Furthermore, Lemma 6.5.8 applies to the sum $A_r(\overline{s}^2c_1, \boldsymbol{c}')$, by construction of r, and observe that $s \ll |\Delta| \ll 1$. As a result we have

$$\begin{aligned}
\sum_{c_1=1}^{\infty} \lambda(c_1)A_q(\boldsymbol{c})I_q(\boldsymbol{c}) &= \sum_{\sigma \pmod{s}} A_s(\overline{r}^2\sigma, \boldsymbol{c}') \sum_{c_1 \equiv \sigma \pmod{s}} \lambda(c_1)A_r(\overline{s}^2c_1, \boldsymbol{0})I_q(\boldsymbol{c}) \\
&= \sum_{\sigma \pmod{s}} A_s(\overline{r}^2\sigma, \boldsymbol{c}') \sum_{c_1 \equiv \sigma \pmod{s}} \lambda(c_1)A_r(c_1, \boldsymbol{0})I_q(\boldsymbol{c}) \\
&= \sum_{\sigma \pmod{s}} A_s(\overline{r}^2\sigma, \boldsymbol{c}')\Sigma_r(\sigma, s),
\end{aligned}$$

say. We apply Proposition 6.5.7 to $A_r(c_1, \boldsymbol{0}) = S_r(c_1)$; let θ and κ be as in the statement of the proposition. Then $\theta, \kappa \mid r$, and we have

$$\begin{aligned}
\Sigma_r(\sigma, s) &= s_r(F) \sum_{\substack{\theta c_1 \equiv \sigma \pmod{s} \\ (c_1, \kappa)=1}} \lambda(\theta c_1)I_q(\theta c_1, \boldsymbol{c}') \\
&= s_r(F) \sum_{\substack{c_1 \equiv \overline{\theta}\sigma \pmod{s} \\ (c_1, \kappa)=1}} \lambda(\theta c_1)I_q(\theta c_1, \boldsymbol{c}').
\end{aligned}$$

Clearing denominators, and using multiplicative characters to cut out the congruence condition $c_1 \equiv \overline{\theta}\sigma \pmod{s}$, we see that

$$\Sigma_r(\sigma, s) = \frac{s_r(F)}{\varphi(\hat{s})} \sum_{\chi \pmod{\hat{s}}} \overline{\chi}(\overline{\theta}\widetilde{\sigma}) \sum_{(c_1, \kappa)=1} \chi(c_1)\lambda((\sigma, s)\theta c_1)I_q((\sigma, s)\theta c_1, \boldsymbol{c}'), \tag{6.45}$$

where $\hat{\sigma} = \sigma/(\sigma, s)$ and $\widetilde{\sigma} = \sigma/(\sigma, s)$. To analyse the inner sum, we need the following

Proposition 6.6.1. *Let χ be a Dirichlet character modulo D, and let θ, κ be positive integers. Then there exists $A > 0$, such that for all $\varepsilon > 0$ we have*

$$\sum_{(n, \kappa)=1} \chi(n)\lambda(\theta n)I_q(\theta n, \boldsymbol{c}') \ll_\varepsilon (\theta\kappa)^\varepsilon D^A \frac{X^{5/6+\varepsilon}}{\theta^{\frac{1}{2}}q^{\frac{1}{3}}}.$$

Proof. Let $S(\chi,\theta)$ be the sum in question. Define the Dirichlet series

$$F_{\chi,\theta}(s) = \sum_{\substack{(n,\kappa)=1}}^{\infty} \frac{\chi(n)\lambda(\theta n)}{n^s}.$$

Since f is a newform,

$$\lambda(mn) = \sum_{d|(m,n)} \mu(d)\lambda(m/d)\lambda(n/d).$$

As a result, for $\sigma > 1$

$$\begin{aligned} F_{\chi,\theta}(s) &= \sum_{\substack{\beta|\theta \\ (\beta,\kappa=1)}} \frac{\mu(\beta)\chi(\beta)\lambda(\theta/\beta)}{\beta^s} \sum_{(n,\kappa)=1} \frac{\chi(n)\lambda(n)}{n^s} \\ &= P(\chi,\theta,\kappa)L(s,f\otimes\chi), \end{aligned} \tag{6.46}$$

where

$$P(\chi,\theta,\kappa) = \prod_{\substack{p^l\|\theta \\ (p,\kappa)=1}} \left(\lambda(p^l) - \frac{\chi(p)\lambda(p^{l-1})}{p^s}\right) \prod_{p|\kappa} \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s}}\right).$$

Recall from (6.4) that $L(s,f\otimes\chi)$ has an Euler product and if χ^* is the primitive character, of conductor D^* say, that induces χ, observe that

$$L(s,f\otimes\chi) = \prod_{p|D/D^*} \left(1 - \frac{\lambda(p)\chi^*(p)}{p^s} + \frac{\chi^*(p)^2}{p^{2s}}\right) L(s,f\otimes\chi^*).$$

Applying (6.5) to $L(s,f\otimes\chi^*)$ for $\frac{1}{2} \le \sigma \le 1$ we get that

$$F_{\chi,\theta}(s) \ll_\varepsilon (b\kappa)^\varepsilon (D^{*2}(1+|t|))^{1-\sigma+\varepsilon}, \tag{6.47}$$

and by (6.6) we get that

$$F_{\chi,\theta}(s) \ll_\varepsilon (\theta\kappa)^\varepsilon (D^{*A}(1+|t|)^{\frac{1}{3}+\varepsilon}, \tag{6.48}$$

when $\sigma = \frac{1}{2}$. Recall the integral $I_q(\boldsymbol{c}', s)$ from (6.9). By the Mellin inversion theorem, we have for any $\sigma > 1$ that

$$S(\chi, \theta) = \frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds.$$

Next, we move the line of integration to $\sigma = \frac{1}{2}$ and use (6.48). To this end, fix $\varepsilon > 0$ and set $T = r^{-1} X^{\varepsilon}$. By (6.10) we have

$$\begin{aligned} \int_{(\sigma)} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds = \int_{\sigma - iT}^{\sigma + iT} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds + \\ O\left(X^{\sigma} r^{-N-1} \int_{|t| \geq T} |t|^{-N}\, dt \right). \end{aligned}$$

The error term is

$$O\left(X^{\sigma} r^{-2} X^{(1-N)\varepsilon} \right).$$

Choosing N large enough we get that

$$\int_{(\sigma)} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds = \int_{\sigma - iT}^{\sigma + iT} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds + O_N(X^{-N}).$$

By (6.47) and (6.10) the horizontal integrals are bounded as follows,

$$\int_{\sigma \pm iT}^{\frac{1}{2} \pm iT} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(s)\, ds \ll X^{\sigma - (N-2)\varepsilon} r^{-\frac{1}{2}}.$$

Once again, choosing N large enough, we get that

$$\int_{(\sigma)} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds = \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds + O_N(X^{-N}).$$

By Lemma 6.4.9, and (6.48) we have

$$\begin{aligned} \int_{\frac{1}{2} - iT}^{\frac{1}{2} + iT} \left(\frac{X}{\theta} \right)^s F_{\chi,\theta}(s) I_q(\boldsymbol{c}', s)\, ds &\ll_{\varepsilon} (\theta\kappa)^{\varepsilon} D^A \left(\frac{X}{\theta} \right)^{\frac{1}{2}} T^{\frac{1}{3}+\varepsilon} \int |I_q(\boldsymbol{c}', s)|\, ds \\ &\ll_{\varepsilon} (\theta\kappa)^{\varepsilon} D^A \left(\frac{X}{\theta} \right)^{\frac{1}{2}} T^{\frac{1}{3}+\varepsilon} \ll_{\varepsilon} (\theta\kappa)^{\varepsilon} D^A \frac{X^{\frac{5}{6}+\varepsilon}}{\theta^{\frac{1}{2}} q^{\frac{1}{3}}}. \end{aligned}$$

This completes the proof of the proposition. $\square$

Applying Propositions 6.6.1 and 6.5.7 to the inner sum in (6.45) we get that

$$\Sigma_r(\sigma, s) \ll_\varepsilon |s_r(F)| X^{\frac{5}{6}+\varepsilon},$$

since $\mathrm{cond}(\chi) \ll s \ll_\Delta 1$. As a result, we have get

$$N^{(0)}(\lambda, \boldsymbol{c}'; X) \ll \sum_{q \ll X} \frac{|s_r(F)| X^{\frac{5}{6}+\varepsilon}}{q^3},$$

where $r \mid q$ is the largest divisor of q that such that $(r; A_1, \ldots, A_4)$ satisfies Condition A_0. Let $q = uv$ where u is square-free and v is square-full. Since $s_q(F)$ is multiplicative, we have by Proposition 6.5.7 that

$$\begin{aligned} N^{(0)}(\lambda, \boldsymbol{c}'; X) &\ll X^{\frac{5}{6}+\varepsilon} \sum_{v \ll X} v^{-\frac{1}{2}} \sum_{u \ll X/v} \frac{1}{u} \\ &\ll_\varepsilon X^{\frac{5}{6}+\varepsilon}, \end{aligned}$$

since the number of square-full $v \le X$ is $O(X^{\frac{1}{2}})$. Summing over $\boldsymbol{c}' \in \mathcal{C}_0$ we obtain the bound

$$N^{(0)}(\lambda; X) \ll_\varepsilon X^{2-\frac{1}{6}+\varepsilon}. \tag{6.49}$$

6.6.2 Contribution from $N^{(1)}(\lambda; X)$

Next we examine $N^{(1)}(\lambda; X)$. For $\boldsymbol{c}' \in \mathcal{C}_1$ define

$$\begin{aligned} N_q^{(1)}(\lambda, \boldsymbol{c}'; X) = \sideset{}{^*}\sum_{a (\mathrm{mod}\ q)} \sum_{\boldsymbol{b} (\mathrm{mod}\ q)} & e_q(aF(\boldsymbol{b}) + \boldsymbol{b}'.\boldsymbol{c}') \\ \times & \sum_{c_1 \equiv b_1 (\mathrm{mod}\ q)} \lambda(c_1) I_q(\boldsymbol{c}). \end{aligned}$$

Then

$$N^{(1)}(\lambda; X) = c_Q X \sum_{\boldsymbol{c}' \in \mathcal{C}_1} \sum_{q \ll X} q^{-3} N_q^{(1)}(F, \lambda, \boldsymbol{c}').$$

By (6.11) and Lemma 6.4.8 we see that

$$N_q^{(1)}(\lambda, \boldsymbol{c}'; X) = \frac{X}{q} \sum_{d|q} \sum_{c_1 \ll X^{1+\varepsilon}/(q/d)^2} \lambda(c_1) S_{d,q}(\boldsymbol{c}) I_{d,q}(\boldsymbol{c}) + O_N(X^{-N}). \tag{6.50}$$

Write $q = uv$ where $(u, 2\Delta) = 1$ is square-free, and v is square-full and is composed of primes dividing $N = 2|\Delta F^{-1}(0, \boldsymbol{c}')|$. Further, decompose $v = v_0 v_1$ where v_0 is the 2^∞-part of v. By Lemma 6.5.12 we have

$$S_{d,q}(\boldsymbol{c}) \ll (u, F^{-1}(0, \boldsymbol{c}')) u^{3+\varepsilon} v_0^{\frac{15}{4}} v_1^{\frac{7}{2}}.$$

Applying Lemma 6.4.10 to estimate $I_{d,q}(\boldsymbol{c})$, obtain the bound

$$\begin{aligned} \sum_{c_1 \ll X^{1+\varepsilon}/(q/d)^2} |\lambda(c_1) I_{d,q}(\boldsymbol{c})| &\ll X^\varepsilon \left(\frac{q}{d^{\frac{1}{2}} X^{\frac{5}{4}}} \right) \sum_{c_1 \ll X^{1+\varepsilon}/(q/d)^2} c_1^{-\frac{1}{4}} \\ &\ll X^\varepsilon \left(\frac{q}{d^{\frac{1}{2}} X^{\frac{5}{4}}} \right) \left(1 + \frac{X^{\frac{3}{4}}}{(q/d)^{\frac{3}{2}}} \right) \\ &\ll \frac{qX^\varepsilon}{X^{\frac{5}{4}}} + \frac{q^{\frac{1}{2}} X^\varepsilon}{X^{\frac{1}{2}}}. \end{aligned}$$

Inserting our bound for $S_{d,q}(\boldsymbol{c})$ into (6.50) we have shown

Proposition 6.6.2. *Suppose that* $1 \leq |\boldsymbol{c}'| \ll X^\varepsilon$. *With notation as above, we have*

$$N_q^{(1)}(\lambda, \boldsymbol{c}'; X) \ll_\varepsilon u^3 v_0^{\frac{15}{4}} v_1^{\frac{7}{2}} \left(\frac{1}{X^{\frac{1}{4}}} + \frac{X^{\frac{1}{2}}}{q^{\frac{1}{2}}} \right) X^\varepsilon.$$

We can also estimate the sum over c_1 in (6.50) using partial summation: employing additive characters to detect the congruence condition $c_1 \equiv b_1 \pmod q$ we have

$$
\begin{aligned}
N_q^{(1)}(\lambda, \boldsymbol{c}'; X) &= \frac{1}{q} \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \sum_{\substack{\boldsymbol{b} \,(\mathrm{mod}\ q) \\ r \,(\mathrm{mod}\ q)}} e_q(aF(\boldsymbol{b}) + b_1 r + \boldsymbol{b}'.\boldsymbol{c}') \sum_{c_1=1}^{\infty} \lambda(c_1) e_q(-rn) I_q(\boldsymbol{c}) \\
&\le \frac{1}{q} \sum_{r \,(\mathrm{mod}\ q)} \Big| \sideset{}{^*}\sum_{a \,(\mathrm{mod}\ q)} \sum_{\boldsymbol{b} \,(\mathrm{mod}\ q)} e_q(aF(\boldsymbol{b}) + b_1 r + \boldsymbol{b}'.\boldsymbol{c}') \Big| \\
&\quad \times \Big| \sum_{c_1=1}^{\infty} \lambda(c_1) e_q(-rc_1) I_q(\boldsymbol{c}) \Big|.
\end{aligned}
$$

By Lemma 6.4.4 we have

$$
\begin{aligned}
\sum_{c_1=1}^{\infty} \lambda(c_1) e_q(-rn) I_q(\boldsymbol{c}) &= - \int \sum_{c_1 \le x} \lambda(c_1) e_q(-rc_1) \frac{\partial}{\partial x} I_q(x, \boldsymbol{c}')\, dx \\
&\ll \frac{(r^{-1}|\boldsymbol{u}'|)^{\varepsilon} |\boldsymbol{u}'|^{-\frac{1}{2}}}{qX} \int_1^X x^{\frac{3}{2}} \log x \, dx + \\
&\quad \frac{(r^{-1}|\boldsymbol{u}'|)^{\varepsilon} |\boldsymbol{u}'|^{-\frac{1}{2}}}{X} \int_1^X x^{\frac{1}{2}} \log x \, dx \\
&\ll_{\varepsilon} \frac{X^{1+\varepsilon}}{q^{\frac{1}{2}}} + q^{\frac{1}{2}} X^{\varepsilon},
\end{aligned}
$$

using the bound

$$
\sum_{n \le z} \lambda(n) e(\alpha n) \ll_f z^{\frac{1}{2}} \log z,
$$

which is uniform in α (see [54, Theorem 5.3]). Recall T_q from (6.42). We have shown

Proposition 6.6.3. *Suppose that $1 \le |\boldsymbol{c}'| \ll X^{\varepsilon}$ and $F^{-1}(0, \boldsymbol{c}') \neq 0$. Then,*

$$
N_q^{(1)}(\lambda, \boldsymbol{c}'; X) \ll_{\varepsilon} \left(\frac{X^{1+\varepsilon}}{q^{\frac{3}{2}}} + \frac{X^{\varepsilon}}{q^{\frac{1}{2}}} \right) T_q.
$$

With Propositions 6.6.2 and 6.6.3 in place, we can complete our analysis of $N^{(1)}(\lambda; X)$. Let $1 \le Y \ll X$ be a parameter to be chosen later. Then

$$
N^{(1)}(F, \lambda, \boldsymbol{c}') = \sum_{q \le Y} q^{-3} N_q^{(1)}(\lambda, \boldsymbol{c}'; X) + \sum_{q > Y} q^{-3} N_q^{(1)}(\lambda, \boldsymbol{c}'; X).
$$

Using Proposition 6.6.2 to estimate the sum up to Y, we get

$$\sum_{q \le Y} q^{-3} N_q^{(1)}(\lambda, \boldsymbol{c}'; X) \ll X^{\frac{1}{2}+\varepsilon} \sum_{v \ll Y} v^{-\frac{7}{2}} v_0^{\frac{15}{4}} v_1^{\frac{7}{2}} \sum_{u \ll Y/v} u^{-\frac{1}{2}} \ll (XY)^{\frac{1}{2}+\varepsilon},$$

since

$$\sum_{\substack{v \ll Y \\ p|v \implies p|N}} 1 \ll_\varepsilon (NY)^\varepsilon.$$

Applying Proposition 6.6.3 to the second sum, we get by Lemma 6.5.10 that

$$\sum_{q > Y} q^{-3} N_q^{(1)}(\lambda, \boldsymbol{c}'; X) \ll_\varepsilon X^{1+\varepsilon} Y^{-\frac{1}{2}} + X^{\frac{1}{2}+\varepsilon}.$$

The optimal choice for Y is $Y = X^{\frac{1}{2}}$, and this gives us

$$N^{(1)}(\lambda; X) \ll_\varepsilon X^{1+\frac{3}{4}+\varepsilon}.$$

Combined with (6.49) this completes the proof of Theorem 1.2.5.

6.7 Deduction of Theorem 6.1.1 from Theorem 1.2.5

This follows by way of a standard argument in moving from estimates for sums with a smooth cut-off to sums with a sharp cut-off. For the sake of brevity, we provide a brief outline of the proof. Let $1 \le P \le X$ be a parameter that we will choose later, and let $\alpha(x)$ be a non-negative, smooth function with support in $[1, X + X/P]$ such that $\alpha(x) = 1$ on $[X/P, X]$ with derivatives satisfying $\alpha^{(j)}(x) \ll_j P^j/x^j$ for all $j \ge 0$. Let $\varepsilon > 0$. Observe that

$$\sum_{m,n \le X} r(Am^2 + Bn^2)\lambda(m) = \sum_{m,n} \alpha(m)\alpha(n)\lambda(m) r(Am^2 + Bn^2) + O_\varepsilon(X^{2+\varepsilon}/P).$$

Applying a smooth partition of unity, it suffices to consider the sum

$$\sum_{m,n} W(m/X)W(n/X)r(Am^2+Bn^2)\lambda(m)$$

with $\mathrm{supp}(W) \in [X/2, X]$, satisfying $W^{(j)}(x) \ll_j P^j/X^j$. Notice that

$$\sum_{m,n} W(m/X)W(n/X)r(Am^2+Bn^2)\lambda(m) = \sum_{\substack{u,v\in\mathbf{Z} \\ Am^2+Bn^2-u^2-v^2=0}} W(m/X)W(n/X)\lambda(m).$$

Applying a smooth partition of unity for the u, v variables, we are left with the sum

$$\sum_{F(\boldsymbol{x})=0} w(X^{-1}\boldsymbol{x})\lambda(x_1),$$

where w is a smooth function with support in $[1/2, 2]$ and with Sobolev norm $\|w\|_{N,1} \ll_N P^N$, and $F(\boldsymbol{x}) = Ax_1^2 + Bx_2^2 - x_3^2 - x_4^2$. It can be shown that the error term in Theorem 1.2.5 depends polynomially on $\|w\|_{1,1}$ — as in the proof of Proposition 5.2 – and we conclude that there exists a constant L such that

$$\sum_{F(\boldsymbol{x})=0} w(X^{-1}\boldsymbol{x})\lambda(x_1) \ll_\varepsilon P^L X^{2-\frac{1}{6}+\varepsilon}.$$

The optimal choice for P[1] is $P = X^{\frac{1}{6(L+1)}-\varepsilon}$, and the theorem follows.

[1]In fact, $L = 9$ is admissible, but we omit the details.

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